THE
MATHEMATICAL GAZETTE

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NOTICE TO NEW MEMBERS.

THE stock of printed copies of the Rules of the Association is now nearly exhausted and the present shortage of paper makes it inadvisable to order a large reprint. It will therefore be necessary to suspend the issue of copies of the Rules to new members. Information about the Rules, as well as any other matters connected with the Association, can be obtained from the Secretaries.

A METHOD OF INVERSION.

BY FRANK H. HUMMEL.

Introduction.

If from a point, regarded as a centre of inversion, a radius is drawn on which are two points at distances r_1 and r_2 from the centre, satisfying the condition $r_1r_2=r^2=\text{constant}$, the points defined by r_1 and r_2 are said to be inverses one of the other.

If with the centre of inversion as centre a circle of radius r is drawn, points on the circle are not changed by inversion and every point inside the circle inverts into a point outside and *vice versa*. Thus the plane may be regarded as divided into two regions with

the circle as a boundary and the constant of inversion r^2 is the same for all directions of rays on which the points lie. If the plane is divided into regions in other ways, the boundary of inversion not being a circle, the constant of inversion will change with the direction of the ray on which the points lie. Thus the boundary of inversion may be any closed figure about the centre of inversion; an ellipse gives simple and interesting properties.

The example considered in this paper is one in which the boundary consists of two parallel lines extended indefinitely, the perpendicular distances of these lines from the centre of inversion being equal.

Method of inversion and general properties.

The system is shown in Fig. 1, in which O is the centre of inversion and $\alpha\alpha$, $\beta\beta$ the boundaries. The inversion formula now is $r_1 r_2 = r^2 \operatorname{cosec}^2 \theta$. Taking O as the origin and rectangular coordinates,

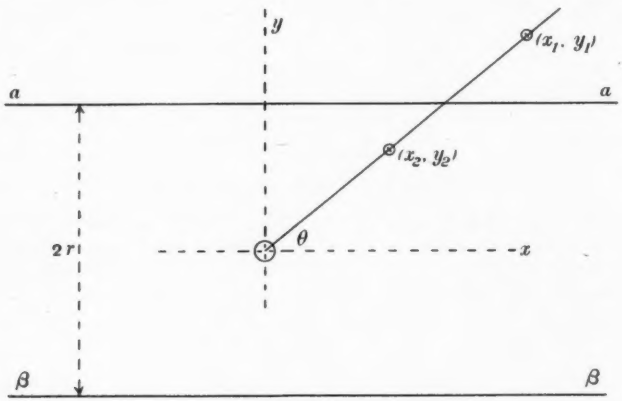


FIG. 1.

$$r_2/y_2 = r_1/y_1 = \operatorname{cosec} \theta;$$
$$r_1 r_2 = r^2 r_1^2/y_1^2 \quad \text{and so} \quad r_2 = r^2 r_1/y_1^2;$$
$$y_2 r_1/y_1 = r^2 r_1/y_1^2 \quad \text{and so} \quad y_2 = r^2/y_1; \quad \text{..............(i)}$$
$$x_2/y_2 = x_1/y_1 \quad \text{and so} \quad x_2 = r^2 x_1/y_1^2. \quad \text{............(ii)}$$

If a curve described in one region is defined by the equation $\phi(x_2, y_2) = 0$, the inverse curve is $\phi(r^2 x_1/y_1^2, r^2/y_1) = 0$.

The curve with the simplest properties in this system of inversion is a parabola with its axis parallel to the boundaries, that is, in this set of axes, parallel to the axis of x.

The following investigation establishes the two fundamental properties of the system.

1. The inverse of a straight line is a parabola with a horizontal axis, passing through the centre of inversion and the points of intersection of the line with the boundaries. The tangent to the parabola at the centre of inversion is parallel to the given line.

2. The inverse of a parabola with a horizontal axis is a parabola with a horizontal axis.

Inversion of a parabola and general properties.

In Fig. 2, O is the centre of inversion and $\alpha\alpha$, $\beta\beta$ the boundaries. Let the given parabola cut the boundaries in P_1, P_2 and the axis of

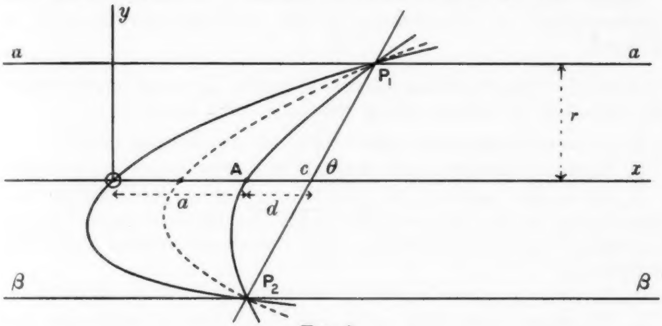


FIG. 2.

x in A. Join P_1, P_2 to cut the axis of x at C, the join making an angle θ with the x-axis. Let $OA = a$, $AC = d$. The equation to the given parabola is

$$x_2 = (d/r^2)y_2^2 + y_2 \cot\theta + a. \qquad \text{(iii)}$$

Substituting for x_2 and y_2 from (i) and (ii) the equation of the inverse figure becomes

$$x_1 = (a/r^2)y_1^2 + y_1 \cot\theta + d. \qquad \text{(iv)}$$

This is the equation to the original parabola with a and d interchanged. It is therefore a parabola passing through P_1 and P_2, cutting the x-axis at a distance d from the centre of inversion. The tangent at the intersection with the axis is parallel to the line P_1P_2 for both the original and the inverse parabolas. The inverse parabola is P_1IP_2 and is shown in the figure by a dotted line. If the centre of inversion is on the original parabola, P_1OP_2 in the figure, $a = 0$ and the inverse figure has the equation

$$x_1 = y_1 \cot\theta + d.$$

This is the equation to the straight line P_1P_2. Hence the properties 1 and 2 above are proved.

One special case of interest is that in which $a = d$. The original and inverse parabolas are then identical.

Problems and geometrical constructions.

The analogy between the results of ordinary circular inversion and the method of parabolic inversion under discussion will be noted. In the former a straight line inverts into a circle passing through the centre of inversion, and a circle inverts into another circle ; in the latter a straight line inverts into a parabola passing through the centre of inversion and a parabola inverts into another parabola. Hence the type of problem involving circles conveniently solved by inversion methods can by this method be solved for parabolas with parallel axes instead of for circles.

Space will not allow of more than one or two examples of such constructions. A classification of the problems can be made as follows.

Series I. Constructions relating to points, lines and one parabola, the direction of the axis of the parabola being known.

1. To draw tangents to a parabola from an external point.
2. To draw a tangent to a parabola at a given point on the curve.
3. To draw a parabola to pass through three given points, one point being on the diameter determined by the other two. This is a standard construction and is a most convenient method of plotting a parabola.
4. To draw a parabola passing through any three given points.
5. To draw a parabola passing through two given points and touching a given line.
6. To draw a parabola passing through a given point and touching two given lines.
7. To draw a parabola touching three given lines.
8. To draw a common tangent to two parabolas.

One example of this series is shown in Fig. 3 ; it is problem No. 5 in the list above.

Given two points P_1 and P_2 and a straight line TT; to draw a parabola (with horizontal axis) to pass through P_1, P_2 and to touch TT.

Take P_2 as centre of inversion and one boundary $\alpha\alpha$ through P_1 (parallel to the given direction of the axis of the parabola) and draw the other boundary $\beta\beta$ parallel to $\alpha\alpha$ and equidistant from P_2. The line TT inverts into the parabola $R_1S_1P_2S_2$. Draw a tangent P_1R_1 to this parabola (Problem 1). The point of contact R_1 inverts into Q_1 on the line TT. The required parabola $P_1Q_1P_2$ passes through P_1 and P_2 and touches the line at Q_1. It can therefore be drawn by the construction of Problem 3. The other tangent from P_1 to the parabola $R_1S_1P_2S_2$ gives another point of contact R_2 which inverts into another point Q_2 on the line TT. Hence a second parabola fulfilling the conditions can be drawn ; it is omitted from the figure for the sake of clarity.

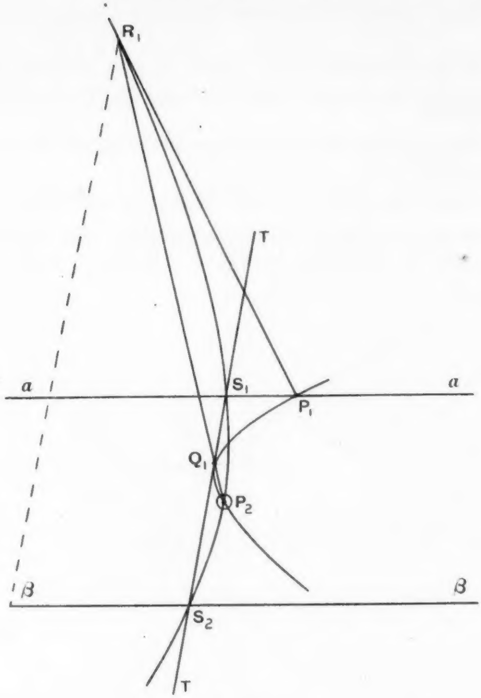


FIG. 3.

Series II. Constructions relating to points, lines and more than one parabola, the directions of the axes of the parabolas being parallel.

1 (*a*). To draw a parabola intersecting a given parabola at a given angle and passing through a given point.

1 (*b*). To draw a parabola touching a given parabola at a point and passing through a given point.

2. To draw a parabola passing through two given points and touching a given parabola.

3 (*a*). To draw a parabola to touch a given parabola and to touch a given line intersecting the parabola, and to pass through a given point.

3 (*b*). To draw a parabola to touch a given parabola and to touch a given line not intersecting the parabola, and to pass through a given point.

4. To draw a parabola to touch a given parabola and two given lines.

5. Given two parabolas and a point on one of them; to draw a parabola to touch the two parabolas, one point of contact being the given point.

6. To draw a parabola touching two given parabolas and passing through a given point.

7. To draw a parabola to touch three given parabolas.

Three examples of these constructions follow. The first is Problem 2 above, and is illustrated in Fig. 4. Points A and B and the

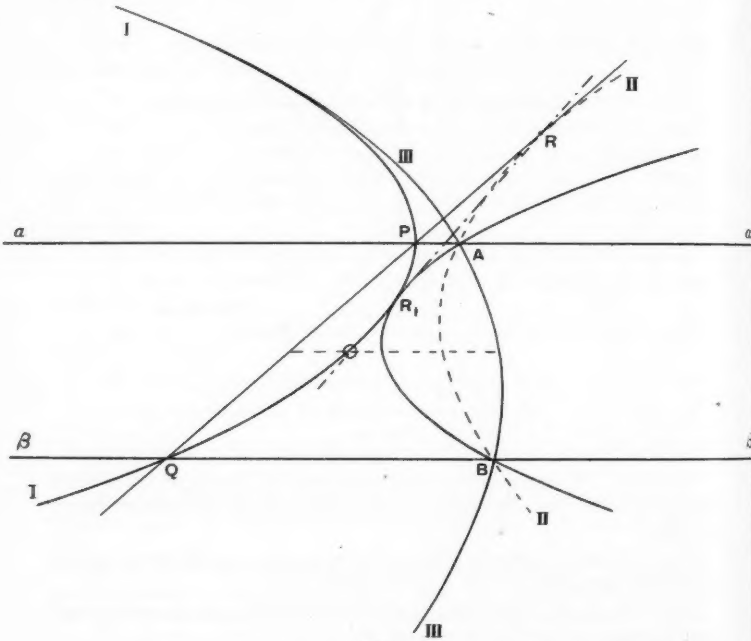


Fig. 4.

parabola I are given, to draw a parabola passing through A and B and touching I.

Bisect AB and draw a horizontal line (*i.e.* a line parallel to the axis of I) to intersect I in O, which we take as the centre of inversion. Draw the boundaries $\alpha\alpha$ and $\beta\beta$, through A and B respectively. The inverse of I is the line PQ. By Problem 5 of Series I draw the parabola II passing through A and B and touching the line PQ at

R. The inverse of R is R_1 and the required parabola passes through A and B and touches I at R_1. A second parabola III can also be found satisfying the conditions. Its construction is not shown in the figure.

The second construction illustrates Problem 3 (b) and is shown in Fig. 5.

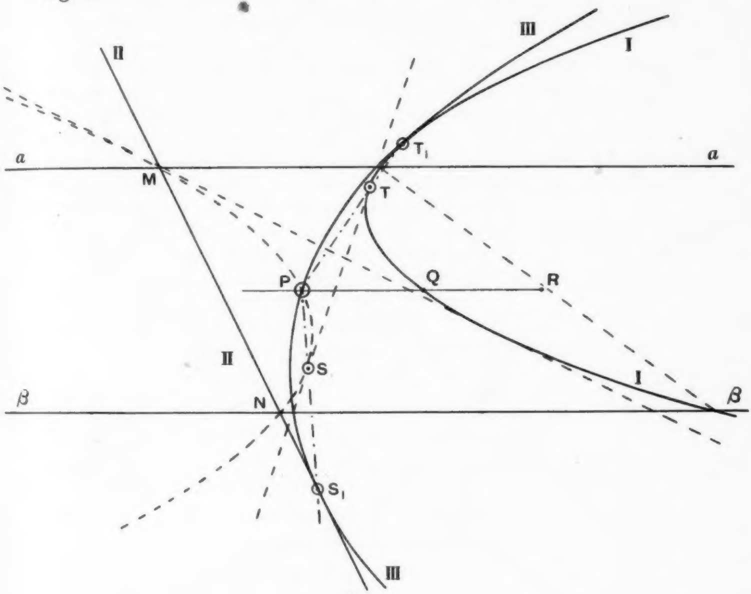


FIG. 5.

A parabola I, a line II and a point P are given; to draw a parabola with a horizontal axis to touch the parabola and the line and to pass through the point P.

Take P as centre of inversion and draw PR, parallel to the axis of I, intersecting I in Q. Take $QR=PQ$ and draw a chord through R parallel to the tangent at Q. The intersections of this chord with I give the boundaries $\alpha\alpha$ and $\beta\beta$. Invert the line II through the points M, N into the parabola MPN. Draw a common tangent ST to the parabolas I and MPN, by Problem 8, Series I. Invert the points of contact S and T to S_1 and T_1. The required parabola touches the line at S_1, passes through P and touches the parabola I at T_1. It is marked III. The second common tangent to I and MPN gives another solution not shown in the figure.

The third construction is Problem 7 and is illustrated in Fig. 6.

Three parabolas I, II and III are given; to draw a parabola to touch these three.

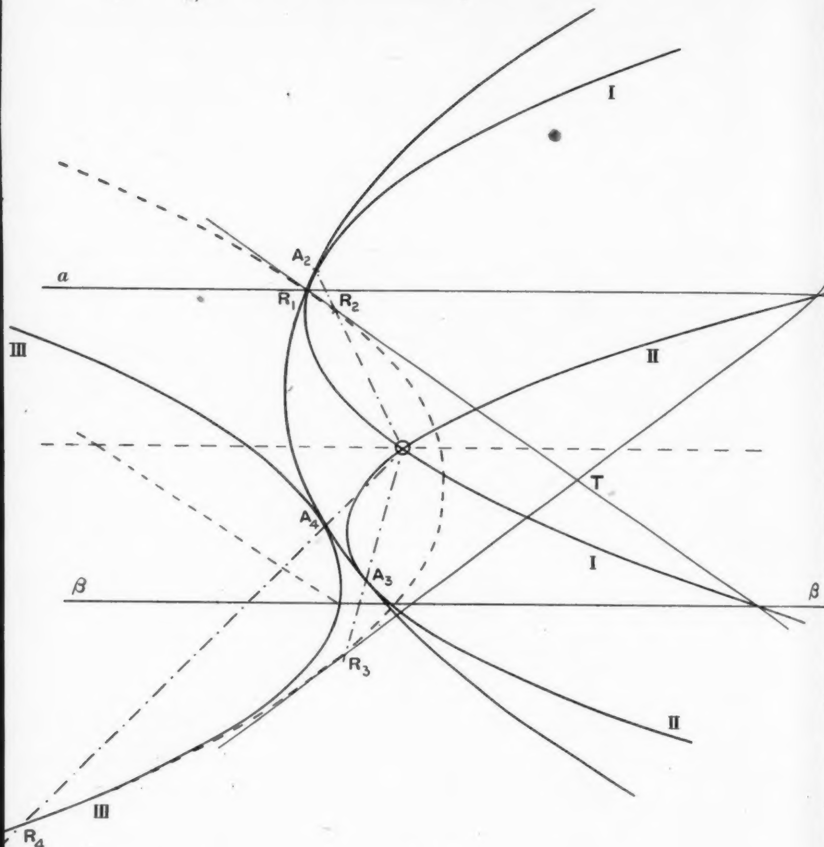


FIG. 6.

Take the centre of inversion at the intersection of *I* and *II*, and any convenient boundaries $\alpha\alpha$ and $\beta\beta$. Parabolas *I* and *II* invert into two lines intersecting at *T*. The parabola *III* inverts into another parabola. (In the figure the boundaries have been so chosen that *III* is its own inverse.) Draw the parabola $R_1R_2R_3R_4$ touching the lines at R_2 and R_3 and the parabola *III* at R_4 (Problem 4). Invert R_2 to A_2, R_3 to A_3 and R_4 to A_4. The required parabola touches *II* at A_3, *III* at A_4 and *I* at A_2.

Corresponding problems concerning the ellipse can be solved by taking an elliptic boundary of inversion. F. H. H.

THE NON-EQUILATERAL MORLEY TRIANGLES.

BY N. M. GIBBINS.

1. In the *Gazette*, No. 248 (February, 1938), pp. 50-57, Mr. W. J. Dobbs gave a very interesting and complete descriptive account of the 18 equilateral " Morley " triangles of a given triangle. The article is especially memorable for its diagrams (1-8) and the present reader is asked to have them at hand for reference.

In the *Gazette*, No. 256 (October, 1939), pp. 367-372, Professor Loria proceeded to find the lengths of the sides of these triangles. Those which he worked out in the text are correct, but many of the results given in the table on p. 372 are wrong, and an amended table is given below. In this, angles only are recorded ; for if the angles against a triangle are θ, ϕ, ψ its side is $8R \sin\theta \sin\phi \sin\psi$. Also in the table and throughout we have written $\frac{1}{3}\pi \equiv \kappa$.

In his article Mr. Dobbs relied on the fact that if two sides and the included angle of a triangle are $\lambda \sin\theta$, $\lambda\sin\phi$, $\pi-\theta-\phi$, then the angles opposite the sides are θ and ϕ respectively. We can go on to say that the third side is $\lambda\sin(\theta+\phi)$.

Then at the top of p. 52 we read that

$$C\xi = 4d \sin\alpha \sin(\kappa-\alpha)\sin(\kappa-\beta) \; ;$$

and $$C\eta = 4d\sin\beta\sin(\kappa-\beta)\sin(\kappa-\alpha),$$

while $$\pi-\alpha-\beta = 2\kappa+\gamma = \pi-(\kappa-\gamma).$$

Hence $$\xi\eta = 4d\sin(\kappa-\gamma)\sin(\kappa-\alpha)\sin(\kappa-\beta).$$

Again, in Fig. 6, mark in z', the point of intersection of Bz and AZ. The angles of ABz' are $\kappa+\alpha$, β, $\kappa+\gamma$.

Hence $$\begin{aligned} Bz' &= d\sin 3\gamma\sin(\kappa+\alpha)/\sin(\kappa+\gamma) \\ &= 4d\sin\gamma\sin(\kappa+\alpha)\sin(2\kappa+\gamma). \end{aligned}$$

Also $$Bx = 4d\sin\gamma\sin(\kappa+\alpha)\sin\alpha \; ;$$

while $$xBz' = \beta = \pi-\alpha-(2\kappa+\gamma).$$

Hence $$xz' = 4d\sin\gamma\sin(\kappa+\alpha)\sin\beta.$$

xyz	α	β	γ	$x'\eta'Z'$	$\kappa-\alpha$	$\kappa+\beta$	γ
$\xi\eta\zeta$	$\kappa-\alpha$	$\kappa-\beta$	$\kappa-\gamma$	$X''\eta''z''$	α	$\kappa+\beta$	$\kappa-\gamma$
XYZ	$\kappa+\alpha$	$\kappa+\beta$	$\kappa+\gamma$	$x''Y''\zeta''$	$\kappa-\alpha$	β	$\kappa+\gamma$
$xy''z'$	$\kappa+\alpha$	β	γ	$\xi''y''Z''$	$\kappa+\alpha$	$\kappa-\beta$	γ
$x'yz''$	α	$\kappa+\beta$	γ	$xX'X''$	κ	α	$\kappa+\alpha$
$x''y'z$	α	β	$\kappa+\gamma$	$yY'Y''$	κ	β	$\kappa+\beta$
$\xi\eta''\zeta'$	α	$\kappa-\beta$	$\kappa-\gamma$	$zZ'Z''$	κ	γ	$\kappa+\gamma$
$\xi'\eta\zeta''$	$\kappa-\alpha$	β	$\kappa-\gamma$	$\xi x'x''$	κ	α	$\kappa-\alpha$
$\xi''\eta'\zeta$	$\kappa-\alpha$	$\kappa-\beta$	γ	$\eta y'y''$	κ	β	$\kappa-\beta$
$XY''Z'$	$\kappa-\alpha$	$\kappa+\beta$	$\kappa+\gamma$	$\zeta z'z''$	κ	γ	$\kappa-\gamma$
$X'YZ''$	$\kappa+\alpha$	$\kappa-\beta$	$\kappa+\gamma$	$X\xi'\xi''$	κ	$\kappa-\alpha$	$\kappa+\alpha$
$X''Y'Z$	$\kappa+\alpha$	$\kappa+\beta$	$\kappa-\gamma$	$Y\eta'\eta''$	κ	$\kappa-\beta$	$\kappa+\beta$
$X'y'\zeta'$	α	$\kappa-\beta$	$\kappa+\gamma$	$Zz'z''$	κ	$\kappa-\gamma$	$\kappa+\gamma$
$\xi'Y''z'$	$\kappa+\alpha$	β	$\kappa-\gamma$				

2. At the end of his article Mr. Dobbs states that 9 of the Morley triangles are *not* equilateral, and that 9 triangles got from corresponding " x ", " y " and " z " vertices *are* equilateral. It is proposed now to establish some properties of the 9 scalene Morley triangles and also of other scalene triangles formed by corresponding vertices " x ", etc., for example, $x\xi X$.

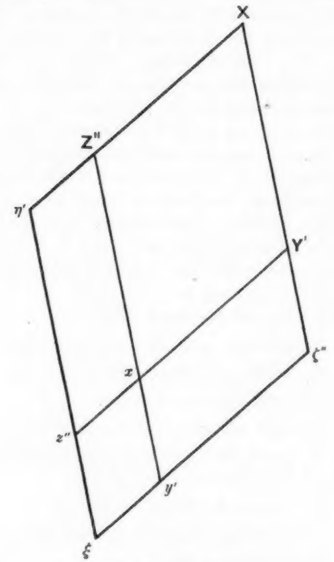


FIG. 9.

Fig. 9 is part of Fig. 8 (of Mr. Dobbs' article) but not drawn to scale, the property of parallelism only being relevant at this stage. By joining X to the mid-point of $x\xi$, η' to the mid-point of $y'Y'$, Z'' to the mid-point of $z''\zeta''$, and using proportionate division, it follows that the following six triangles have the same centroid :

$$Xx\xi,\ Xy'z''\ ;\ \ \eta'y'Y',\ \eta'x\zeta''\ ;\ \ Z''z''\zeta'',\ Z''\xi Y'.$$

These may be written in the following order :

$$x\xi X,\ y'\eta'Y',\ z''\zeta''Z''\ ;\ \ x\eta'\zeta'',\ \xi Y'Z'',\ Xy'z'',$$

the first three of which are scalene triangles of corresponding vertices, and the last three scalene Morley triangles.

Similarly by extracting two other portions of Fig. 8, by cyclic changes of the letters, beginning with $y\eta Y$ and $z\zeta Z$, we have, further, that the following two sets of six triangles have respectively a common centroid, the same classification as above being applicable :

$$y\eta Y,\ z'\zeta'Z',\ x''\xi''X''\ ;\quad y\zeta'\xi'',\ \eta Z'X'',\ Yz'x''\ :$$
$$z\zeta Z,\ x'\xi'X',\ y''\eta''Y''\ ;\quad z\xi'\eta'',\ zX'Y'',\ Zx'y''.$$

3. We now proceed to find a construction for the centroid of $x\xi X$ in terms of ABC.

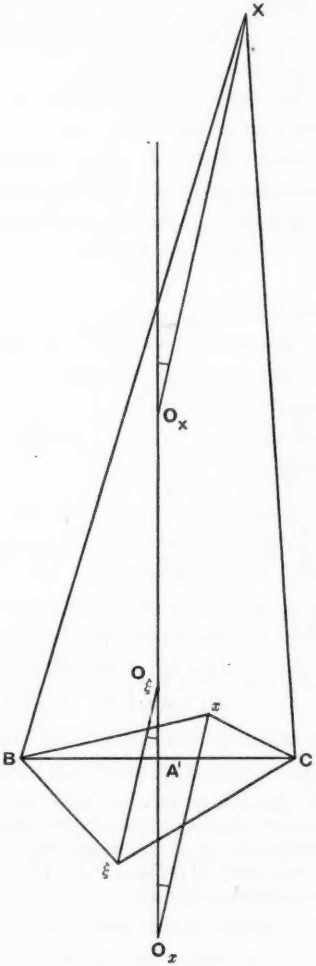


FIG. 10.

Let O_x, R_x; O_ξ, R_ξ; O_X, R_X be the circumcentres and circumradii of the triangles xBC, ξBC, XBC; and A' the mid-point of BC. The angles of these triangles are, respectively,

$$2\kappa+\alpha,\ \beta,\ \gamma;\quad 2\kappa-\alpha,\ \kappa-\beta,\ \kappa-\gamma;\quad \alpha,\ \kappa+\beta,\ \kappa+\gamma.$$

If $\gamma > \beta$, the angles which $O_x x$, $O_\xi \xi$, $O_X X$ make with the perpendicular bisector of BC are $\gamma-\beta$, $(\kappa-\beta)-(\kappa-\gamma)$, $(\kappa+\gamma)-(\kappa+\beta)$ respectively, and these lines are therefore parallel.

We have also that

$$BC = 2R_x \sin(2\kappa+\alpha) = 2R_\xi \sin(2\kappa-\alpha)$$
$$= 2R_X \sin\alpha = 2R \sin 3\alpha.$$

Hence $\quad R_x = 4R\sin\alpha\sin(\kappa+\alpha) = R\{1-2\cos(\kappa+2\alpha)\},$

$$R_\xi = 4R\sin\alpha\sin(\kappa-\alpha) = R\{2\cos(\kappa-2\alpha)-1\},$$
$$R_X = 4R\sin(\kappa-\alpha)\sin(\kappa+\alpha) = R(1+2\cos 2\alpha),$$

so that $$R_x - R_\xi + R_X = 3R.$$

Again,

$$O_xA' = R_x\cos(2\kappa+\alpha)$$
$$= -R\{\cos(\kappa-\alpha) - 2\cos(\kappa-\alpha)\cos(\kappa+2\alpha)\}$$
$$= -R\{2\cos(\kappa-\alpha)-\cos 3\alpha\},$$
$$O_\xi A' = R_\xi\cos(2\kappa-\alpha)$$
$$= R\{\cos(\kappa+\alpha) - 2\cos(\kappa+\alpha)\cos(\kappa-2\alpha)\}$$
$$= R\{2\cos(\kappa+\alpha)-\cos 3\alpha\},$$
$$O_XA' = R_X\cos\alpha$$
$$= R(\cos\alpha + 2\cos\alpha\cos 2\alpha)$$
$$= R(2\cos\alpha+\cos 3\alpha).$$

Taking $A'C$ and $A'O_X$ as axes of coordinates, we have the table:

x	$R_x\sin(\gamma-\beta)$,	$R_x\cos(\gamma-\beta) + R\{\cos 3\alpha - 2\cos(\kappa-\alpha)\}$;
ξ	$-R_\xi\sin(\gamma-\beta)$,	$-R_\xi\cos(\gamma-\beta) + R\{\cos 3\alpha - 2\cos(\kappa+\alpha)\}$;
X	$R_X\sin(\gamma-\beta)$,	$R_X\cos(\gamma-\beta) + R(\cos 3\alpha + 2\cos\alpha)$.

The coordinates of the centroid of $x\xi X$ are therefore

$$R\sin(\gamma-\beta),\ R\cos(\gamma-\beta) + R\cos 3\alpha.$$

Hence draw OP parallel to $O_x x$ to meet the circle ABC in P. Then P is the centroid of $x\xi X$. Since OA makes an angle $3\gamma-3\beta$ with $A'O_X$, PA subtends at O an angle $2(\gamma-\beta)$. Similarly the centroid of $y\eta Y$ is at a point Q on the circle ABC, and if $\gamma > \alpha$, then QB subtends at O an angle $2(\gamma-\alpha)$.

In Fig. 11, $\quad POQ = AOB - POA - QOB$

$$= 6\gamma - 2(\gamma-\beta) - 2(\gamma-\alpha)$$
$$= 2(\alpha+\beta+\gamma) = \tfrac{2}{3}\pi.$$

Also PQ makes with BC an angle

$$\tfrac{1}{2}\pi - \tfrac{1}{6}\pi - (\gamma - \beta) = \alpha + \beta + \gamma - \gamma + \beta = \alpha + 2\beta,$$

and PQ is therefore parallel to xy.

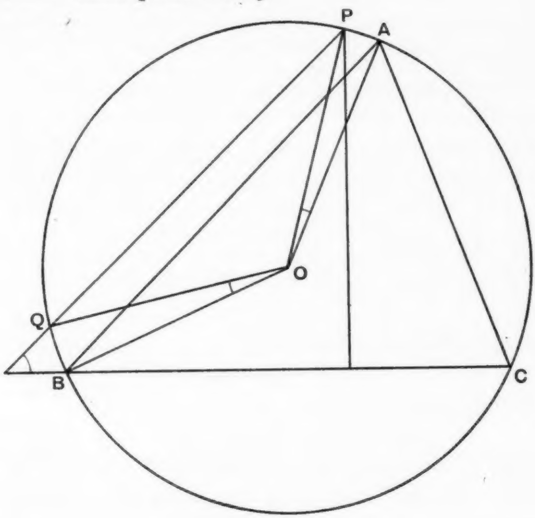


Fig. 11.

Finding the centroid R of $z\zeta Z$ in the same way, it follows that P, Q, R are on the circle ABC, and are the vertices of an equilateral triangle whose sides are parallel to those of all the equilateral triangles of Fig. 8.

Referring to § 2, we accordingly have the result that the nine scalene Morley triangles can be divided into sets of three, each set having a common centroid, the three centroids being spaced as above. The same thing is true of the following three sets of triangles of corresponding vertices :

$$x\xi X,\ y'\eta' Y',\ z''\zeta'' Z''\ ;\ y\eta Y,\ z'\zeta' Z',\ x''\xi'' X''\ ;\ z\zeta Z,\ x'\xi' X',\ y''\eta'' Y''.$$

4. Referring to Fig. 9, and denoting the area of the parallelogram $xY'XZ''$ by (Xx), etc., we have

$$2x\eta'\zeta'' = (xX) - (x\xi),$$
$$2\xi Y'Z'' = (\xi X) - (\xi x),$$
$$2Xy'z'' = (X\xi) - (Xx)\ ;$$

so that $$\xi Y'Z'' = x\eta'\zeta'' + Xy'z''.$$

Similarly $$\eta Z'X'' = \eta\zeta'\xi'' + Yz'x'',$$

and $$\zeta X'Y'' = z\zeta'\eta'' + Zx'y''.$$

To calculate (Xx) we have (*see* Fig. 8 and the table of § 1):

$$XY' = XZ - Y'Z = 8R\sin(\kappa+\alpha)\sin(\kappa+\beta)\{\sin(\kappa+\gamma)-\sin(\kappa-\gamma)\},$$

$$XZ'' = XY - YZ'' = 8R\sin(\kappa+\alpha)\sin(\kappa+\gamma)\{\sin(\kappa+\beta)-\sin(\kappa-\beta)\}.$$

Hence

$$(Xx) \equiv XY' \,.\, XZ''\sin\kappa$$
$$= 32\sqrt{3} \,.\, R^2\sin^2(\kappa+\alpha)\sin(\kappa+\beta)\sin\beta\sin(\kappa+\gamma)\sin\gamma.$$

Again

$$X\eta' = XZ' - \eta'Z' = 8R\sin(\kappa-\alpha)\sin(\kappa+\beta)\{\sin(\kappa+\gamma)-\sin\gamma\},$$

$$X\zeta'' = XY'' - \zeta''Y'' = 8R\sin(\kappa-\alpha)\sin(\kappa+\gamma)\{\sin(\kappa+\beta)-\sin\beta\}.$$

Hence

$$(X\xi) \equiv X\eta' \,.\, X\zeta'' \,.\, \sin\kappa$$
$$= 32\sqrt{3} \,.\, R^2\sin^2(\kappa-\alpha)\sin(\kappa+\beta)\sin(\kappa-\beta)\sin(\kappa+\gamma)\sin(\kappa-\gamma).$$

Hence

$$Xy'z'' = 16\sqrt{3} \,.\, R^2\sin(\kappa+\beta)\sin(\kappa+\gamma)$$
$$\{\sin^2(\kappa-\alpha)\sin(\kappa-\beta)\sin(\kappa-\gamma)-\sin^2(\kappa+\alpha)\sin\beta\sin\gamma\}$$
$$= 6R^2\sin\alpha\sin(\kappa+\beta)\sin(\kappa+\gamma)$$
$$\{1-8\cos\alpha\cos(\kappa+\beta)\cos(\kappa+\gamma)\}.$$

It is easy to prove from the coordinates for x and ξ in § 3 that

$$x\xi^2 = 12R^2\sin^2\alpha\,\{1-8\cos\alpha\cos(\kappa+\beta)\cos(\kappa+\gamma)\};$$

and it may be proved in the same way, or from the triangle ABX, that

$$AX^2 = 16R^2\sin^2(\kappa+\beta)\sin^2(\kappa+\gamma)\{1-8\cos\alpha\cos(\kappa+\beta)\cos(\kappa+\gamma)\}.$$

Hence $\quad x\xi \,.\, AX = 8\sqrt{3}R^2\sin\alpha\sin(\kappa+\beta)\sin(\kappa+\gamma)$

$$\{1-8\cos\alpha\cos(\kappa+\beta)\cos(\kappa+\gamma)\},$$

so that $\quad Xy'z'' = \tfrac{1}{4}\sqrt{3} \,.\, x\xi \,.\, AX;$

similarly $\quad \xi Y'Z'' = \tfrac{1}{4}\sqrt{3} \,.\, xX \,.\, A\xi,$

and $\quad x\eta'\zeta'' = \tfrac{1}{4}\sqrt{3} \,.\, \xi X \,.\, Ax.$

By cyclic changes of the letters in the above formulae we may write down the areas of the remaining scalene Morley triangles.

5. Since (§ 4) $\xi Y'Z'' = x\eta'\zeta'' + Xy'z''$, we have from the above

$$xX \,.\, A\xi = \xi X \,.\, Ax + x\xi \,.\, AX.$$

Hence the circle $x\xi X$ passes through A.

Similarly y, η, Y, B and z, ζ, Z, C are concyclic.

We may also show that the circles $x'\xi'X'$ and $x''\xi''X''$ pass through A, with similar results for the other circles in relation to B and C. It will be found that the "x" circles are co-axal, as are the "y" and "z" circles, the radical axes being the medians of ABC; so that the centroid of ABC is the radical centre of the nine circles.

N. M. G.

PRINCIPLES OF THEORETICAL PHYSICS.*

BY D. K. PICKEN.

1. *Magnitude.*

PHYSICAL Magnitude is of a variety of " kinds " : not of only one or two kinds ; nor, on the other hand, of an infinity—or an unlimited number—of kinds ; but of a very considerable (though not perhaps quite definite) number of kinds. A catalogue of these—not set out for just that purpose—is to be found on pp. 3-7 of Kaye and Laby (1st Edn.).†

The distinguishing characteristic of " magnitude " is use of the comparatives " greater " and " less ", in some specific sense. The variety of ways in which we use these comparatives—as a matter of common sense (more or less refined)—determines the variety of the " kinds of magnitude ". Thus we say greater or less *in Length* ; *in Area* ; *in Volume* ; *in Duration* ; *in Speed* ; *in Mass* ; *in Weight* —and so on. But it would be meaningless to say (*e.g.*) that the *area* of a (closed) plane figure is greater than the *length* of some line in it ; or that the *mass* of a body is greater than its *volume* ; or that the *mass* of one body is greater than the *speed* of motion of another—or suchlike.

2. *Quantities.*

The question, " *How much greater (or less)?* "—in any particular (legitimate) case of comparison—is the beginning of Physics as an " *exact* ", rather than merely a " descriptive ", *science*. There are two different modes of answer to the question, viz. :

(i) in terms of " difference ", (ii) in terms of " ratio ".

With the former of these is bound up the conception of " sum ", in this context ; with the latter, that of " measurement ".

And what we so " subtract "—or " add "—or " measure " are called " quantities " : the Quantities of Physics (with which are incorporated those of Geometry). For each " kind of magnitude " there is a system of quantities (" the quantities of that kind ") : *lengths* ; *angles* ; *areas* ; *volumes* ; ...... ; *times* ; *speeds* ; ...... ; *masses* ; *forces* ; .... And, because Magnitude, as we know it in experience, is, in general, " continuous " (basically, in terms of the " continuity " of space and time), there is an infinity of quantities of each kind—in two senses, viz. : (i) of all the quantities within any given finite (continuous) range, (ii) of the infinity of the full range of all the quantities of the kind in question.

* In 1939 the writer delivered a Presidential Address to the Mathematical Association of Victoria, on " The Theoretical Structure of Physics " (see *Gazette*, July, 1940, p. vi.). This was being written up for publication when the outbreak of war interrupted such work. This present paper, on somewhat different lines, was written recently—at the request of Professor T. H. Laby, F.R.S.—for inclusion in the typed Lecture Notes in Natural Philosophy of the University of Melbourne. Some mathematical points elaborated in the Address are referred to briefly in footnotes.

† " Physical and Chemical Constants " by G. W. C. Kaye and T. H. Laby (Longmans, Green & Co.).

Thus there are Physical Quantities of a considerable variety of different kinds; and there is an infinity of quantities of each of these kinds. These are not themselves actual tangible Physical things. They are "abstract" scientific entities: "abstracted" from the Physical facts and phenomena—and only a stage less "abstract" than the entities of Pure Mathematics (with which they are, in fact, closely bound up).

There are principles of simplification—in terms of "idealisation" —in the process of abstraction: marking the dividing line between Theoretical Physics and the Physics of the actual phenomena—and accounting for discrepancies between theoretical results and the actual Physical facts to which they refer (whereby are detected flaws in the process of "abstraction" of quantities and their relations to one another). Quantities belong essentially to the realm of Theoretical Physics; they are "exact" (like the entities of Pure Mathematics); and it is "an exact science"—being connected with the Physics of the phenomena by the methods and processes of Experimental Physics.

Questions as to "How much?"—being necessarily "exact" in type—are essentially applicable to the quantities of Theoretical Physics. The answer to such a question is "*exact*", but not necessarily "true". (It is "true" within the limits of accuracy of the experimental determination of the quantities in question; *i.e.* the "true" answer is known to be between two extremes, which are determinate by correlation of the methods of theory and experiment.)

3. *Specification of Quantities.*

The question *BY how much* one quantity is greater (or less) than another (of the same kind)—a question of "difference"—is not in general answered directly; because the answer to the question is expressed by a third quantity of the same kind, and the whole matter is therefore affected by how, in fact, quantities in general are specified.

The general scientific answer to the question, "*How much more* (or less)?—of quantities of any one kind—is in terms of "ratio"; the *ratio of any two quantities* OF THE SAME KIND being a "number" (in the mathematical sense of that term),* by means of which the first of the quantities is specifiable in terms of the second. All the quantities of any one kind are thus specifiable, in terms of any one of them, by means of the "numbers" which are their ratios to it; and, as the number which then specifies *it* is 1, it is called the "*unit* quantity" of the specification. The process of determining these ratios is called "Measurement"; and the numbers so determined are called the "measures" of the several quantities, in terms of the chosen "unit".

The usefulness of this mode of specification is indicated by

* See *Gazette*, July, 1938, xxii, 250, pp. 225-233.

the following simple propositions (general proofs not quite simple *) :

(i) *The measure of the difference* (or of the *sum*) *of any two of the quantities = the difference* (or the *sum*) *of their measures.*

(ii) *The ratio of any two of the quantities = the quotient of their measures.*

(Symbolically, if $P : U = p$, $Q : U = q, \ldots$, denote the several measures (relations also written $P = p \,.\, U$, $Q = q \,.\, U, \ldots$), then

$$\text{(i)}\ (P \pm Q) : U = p \pm q\ ; \quad \text{(ii)}\ P : Q = p/q.)$$

4. *Length and Angle : the Fundamental " Kinds ".*

Actually, Addition-and-Subtraction and Measurement are applicable directly to only the two fundamental geometrical types, *Length* and *Angle.*

Addition of *lengths* is according to the relation $AB + BC = AC$, for collinear points A, B, C, in that order ; and *subtraction*, accordingly. And the investigation of *ratio of lengths* (as by the ancient Greeks) is the common-sense basis of the (somewhat difficult) theory of the " numbers " of Pure Mathematics—thence fundamental to all theory of Measurement (of " physical quantities ", by " numbers "). This investigation uses " multiples " ($m \,.\, P = P + P + \ldots$; m integral) or " sub-multiples " ($P/n = Q$, such that $n \,.\, Q = P$; n integral) ; it is theoretically " exact ", in terms of " the Real Numbers " of Pure Mathematics ; and practically appropriate, in terms of " rational (or decimal) approximations ", to any required degree of accuracy. (Note that " approximation " has always a reference back to " exactness ").†

The case of *angles* is closely analogous : from the addition relation $\angle AOB + \angle BOC = \angle AOC$ (OA, OB, OC coplanar in that order) ; and " number ", as investigated in terms of ratio of lengths, is thus immediately applicable to the measurement of angles. But here there are the distinctive facts (i) of a " natural unit ", viz. the (so-called) " angle of one revolution " (or aliquot part thereof, such as straight angle, right angle, degree) ; (ii) of a second " absolute " mode of measurement of *angles*, viz. " circular measure " of angles.‡

5. *Proportion.*

On the foundation of these two fundamental types is raised (as by the Greeks) a theory of " *Proportion* "—which is a type of relation between quantities of (in general) DIFFERENT *kinds* (involving at least two of each kind). P, Q, of one kind, are said to be *proportional* to X, Y of another (or, it may be, the same) kind, if $P : Q = X : Y$; *inversely proportional*, if $P : Q = Y : X = 1/(X : Y)$.

* Because they necessarily involve the theory of the Real Numbers; but the practically important cases of p, q integral or fractional numbers are comparatively simple.

† xxii, 250, p. 229, § 4.

‡ See footnote to § 5, *infra*, on the fundamental theorems of Joint Proportion.

More generally, functional relations of proportion between *variable* (*i.e.* representative) quantities P, X (in general, but not necessarily, of different kinds)—if the ratio of *any two* of the P-quantities = the ratio of the *two corresponding* X-quantities (direct or inverse).

And, from the fundamental geometrical cases (involving *lengths* and *angles*—and *areas*),* there emerges the type of relationship, *between more than two variable quantities*, known as "Joint Proportion"—direct and inverse; the simplest case being: P jointly proportional (directly) to X and V, where P, X, V are functionally related (*e.g. area* of rectangle, and *lengths* of its sides) in such a way that P is proportional to X, when V is not varied, and P is proportional to V, when X is not varied †; a relation from which is deduced the proposition

$$(P_r : P_s) = (X_r : X_s) \,.\, (V_r : V_s)$$

relating ratios of any *corresponding* pairs of the quantities concerned.‡

The general case of Joint Proportion, direct and inverse, gives the type of proposition:

$$(P_r : P_s) = \Pi(X_r : X_s)/\Pi(V_r : V_s)$$

where P is *directly* proportional to each X, when all the quantities other than these two (P and that X) are not varied; *inversely* proportional to each V, similarly. *This is the central proposition of the whole theory.*

6. *Theoretical Structure.*

The fundamental theoretical structure of Physical Science is a structure of relations of joint proportion, *together with analogous relations of the more general type expressed by*

$$(P_r : P_s) = \Pi((X_r : X_s)^h),$$

* The fundamental theoretical propositions of Proportion are, of course, geometrical and are bound up with the fundamental relation of "number" (as ratio) to both Length and Angle. They are: (1) The proportionality of the (variable) *lengths OP, OQ*, if *PQ* remains constant in direction; (2) the proportionality of arc-*length* to *angle*, for sectors of a given circle; (3) the proportionality of arc-*length* to radius (*length*), for sectors of given angle; (4) the proportionality of *area* to base-*length*, for triangles (or parallelograms) of given altitude.

† The basic case of Joint Proportion (in terms of the basic kinds of Magnitude, Length and Angle) is: (1) arc-*length* of circle-sector jointly proportional to *angle* and radius (*length*); expressible, conversely, in the form: (2) *angle* of sector jointly proportional, *directly* to arc-*length*, *inversely* to radius (*length*). Only secondary to these is: (3) *area* of rectangle jointly proportional to the *two lengths* of the sides. These propositions of Geometry appear to be basic, as the concrete foundation of the whole Proportion-structure of Theoretical Physics. The proposition (2) leads to Circular Measure of Angle; (3), to the "derived unit" of Area. (See § 7, *infra*, for the latter; and § 8, *infra*, with footnote, for the former.)

‡ Tacit to the proof of this type of proposition is the basic assumption (see § 2, *supra*) that the kinds of magnitude in question are "continuous", and that there is a quantity P corresponding to any given set of quantities of the other kinds in question; *e.g.* in the simple cases of the preceding footnote, P' corresponding to X_2 and V_1—so that $(P' : P_1) = (X_2 : X_1)$ and $(P_2 : P') = V_2 : V_1$, or to $V_1 : V_2$; the theorem in question then resulting from $(P_2 : P_1) = (P_2 : P') \times (P' : P_1)$—for which cf. § 3, *supra*, with footnotes.

in which each h is a POSITIVE OR NEGATIVE "rational" number (integral or fractional): relations for which (following Euclid and Newton) the name *joint* "MULTIPLICATE *proportion*" may be used (more specifically, h-plicate).

The actual functional relations of the theory are—practically all—simple cases of these two forms: many of them involving only three variables (P, X, V) and only the simplest integral or fractional values of h (such as ± 1, ± 2, $\pm 3/2$).

The theory, as a whole, is strung upon *a chain of such functional relations*, linked together by quantities of the same kind involved in different relations; as follows: (i) the elementary geometrical relations, between *Length, Angle, Area, Volume,* ......; (ii) the kinematical relations, between *Length, Angle, Time, Velocity, Acceleration,* ......; (iii) the dynamical relations, between *Velocity, Acceleration, Mass, Force, Momentum, Energy,* ......; and so on.

7. "*Derived Units*" *and the Mathematical Theory.*

The chain of functional relationships is used primarily in the form

$$(P : \mathsf{P}) = \Pi((X : \mathsf{X})^h),$$

or

$$p = \Pi(x^h),$$

in terms of "measures" x, p—where the unit P is chosen to "correspond", in the functional relation in question, to units X which have already been chosen (in accordance, so far as possible, with the same principles).

These relations in terms of "measures" (*i.e. numbers*) constitute the groundwork of the mathematical theory of Physics. The units to which they refer are (i) certain basic units, chosen arbitrarily, viz. in the standard system, units of *Length, Time, Mass* (together with certain others, as they subsequently arise); (ii) "derived units", determined from them, as above (*e.g. unit of area,* by means of the functional relation between the variable quantities of *the Rectangle,* as *area of a rectangle which has sides of unit length*; *unit of speed,* by means of the functional relation—of direct and inverse proportion—used in defining speed, as (average) *speed of a motion of unit length and of unit duration,* etc.).

It is to be noted that the use of such a system of basic and derived units is not merely convenient, but is of the very essence of the scientific theory. There is, however, an element of arbitrariness in the choice of the basic types—except *Length,* which is in every sense basic *; and, more particularly, in the choice of *Mass*—for which a derived unit is properly determinate (along with that of *Force*), from basic units of *Length* and *Time,* by means of the *two* functional relations: (i) of Newton's Second Law of Motion, (ii) of Newton's Law of Gravitation. (This DERIVED *unit of Mass* is in use in Astronomical Dynamics and in Relativity theory. It is the *mass of a particle* (or of a spherical body) *to gravitational action of which unit acceleration is due at unit distance*).

* *Time,* as basic, is in actual fact bound up with *Angle,* as geometrically basic. See § 4, *supra* and § 8 (i), *infra,* with footnote.

It is to be noted, further, that if a Physical functional relation of the type in question be such that, for *all* the kinds of magnitude involved in it, the units have been chosen independently of it, the mathematical expression (in terms of " measures ") of the relation takes the form

$$p_r/p_s = \Pi((x_r/x_s)^h) \text{ or } p_r/\Pi(x_r^h) = p_s/\Pi(x_s^h),$$

i.e $p/\Pi(x^h)$ is *constant*,

or $$p = k \,.\, \Pi(x^h)$$

where k does not vary with the X's and P, but does change if the units (X, etc.) involved in the mathematical relation are changed (more particularly, for such change of units as is considered in § 8).

The more scientific the treatment of the theory, the less important does this latter form of mathematical relation * become.

8. *Systematic Change of Units.*

(i) " *Dimensional* " *Numbers.* A second fundamentally important use of the standard relations of § 6 is in determination of the *change of derived units* consequent upon change of the basic units: by means of the form

$$(\mathsf{P}' : \mathsf{P}) = \Pi((\mathsf{X}' : \mathsf{X})^h) = \Pi((\mathsf{X}' : \mathsf{X})^s),$$

where s is either h itself, or, if two or more of the X-quantities involved in the functional relation in question are *of the same kind* is Σh, for these. (In particular, s may be 0—as, *e.g.*, in the functional relation of *angle* to *arc-length* and *radius*, for Circle-Sector.†)

Writing this $$\varpi = \Sigma(\xi^s),$$

and, using in it expressions similarly obtained (from the appropriate functional relations) for each of the ξ's, until we reduce to expressions in $$\lambda(=\mathsf{L}' : \mathsf{L}),\ \tau(=\mathsf{T}' : \mathsf{T}), \text{ etc.},$$

we get $$\varpi = \lambda^a \,.\, \tau^b \dots ,$$

where $a, b, \dots$, are rational numbers, positive or negative.

The exponent numbers $a, b, \dots$, are the (so-called) " *dimensions* " of the unit quantity P (or of that kind of magnitude); *e.g.* the "dimensional " expression for *Area* is λ^{+2}; for *Speed*, $\lambda^{+1} \,.\, \tau^{-1}$; for *Acceleration*, $(\lambda^{+1} \,.\, \tau^{-1}) \,.\, \tau^{-1} = \lambda^{+1} \,.\, \tau^{-2}$; for *Force*, $\lambda^{+1} \,.\, \tau^{-2} \,.\, \mu$, if Mass be regarded as basic—but, if not (as explained in § 7, above), then for *Mass*, $\lambda^{+3} \,.\, \tau^{-2}$, and for *Force*, $\lambda^{+4} \,.\, \tau^{-4}$: etc.

* Which has commonly been taken as the fundamental form for the type of Physical functional relation in question. The technique of §§ 5, 6, *supra*, seems to give true perspective to this theory.

† Here from, say, $A_2 : A_1 = (S_2 : S_1)/(R_2 : R_1)$, since R and S are of the same kind (both *lengths*), the general forms of §§ 7, 8 reduce to

$$(1)\ (A : \mathsf{A}) = (S : \mathsf{L})/(R : \mathsf{L}) = (S : R)$$

—the theorem of Circular Measure, the unit angle A corresponding to $S = R(=\mathsf{L})$, but being in fact independent of the unit length, L; (2) $(\mathsf{A}' : \mathsf{A}) = (\mathsf{L}' : \mathsf{L})^0 = 1$—again expressing that independence.

The case of *Angle*, the second of the two fundamental kinds of magnitude (see § 4), is to be specially noted. For the special reason indicated above, it is properly said to be of dimension 0 in terms of Length (Time, etc., not relevant). This, in fact, leaves it dimensionally " basic "—affecting dimensionally the units of such kinds of magnitude as Angular Velocity, Angular Acceleration, for change of unit angle, *e.g.* from the practical unit, for such a case (viz. *angle of one revolution*), to the theoretical unit (of Circular Measure, *i.e.* *Radian*). But for purely theoretical purposes, the use of circular measure of angles is general—as being proper to the whole theoretical structure ; and no question of change of unit is involved.

The question of the relation of measurement of *Time* to measurement of *Angle* is important—as all continuous measurement of Time is by proportion to angles of rotation or of " revolving ", in uniform motions. (See Kaye and Laby, pp. 3, 4.*)

(ii) *The " Dimensions " Test.*

Physical relations, when expressed *mathematically*—in terms of " measures "—have *general* mathematical forms which do not depend on the actual units in terms of which the Physical quantities are measured, but only upon the fact that these constitute a theoretical " system " of such units (as in § 7, *supra*). *E.g.*, for the fundamental Law of Motion, the general form $f=m\,.\,a$.

It follows that, if in such a form, we substitute

$$l=\lambda\,.\,l',\ t=\tau\,.\,t',\ \dots,\ v=(\lambda^{+1}\,.\,\tau^{-1})\,.\,v',\ \dots,$$

the result must reduce to the same *form*, in the new variables, as in the original variables.

This fact can be used as a check of accuracy. *E.g.*, in a simple case,

$$x=V\,.\,t+\tfrac{1}{2}\,.\,a\,.\,t^2$$

gives $\lambda\,.\,x'=(\lambda^{+1}\,.\,\tau^{-1})\,.\,V'\,.\,\tau\,.\,t'+\tfrac{1}{2}\,.\,(\lambda^{+1}\,.\,\tau^{-2})\,.\,a'\,.\,(\tau\,.\,t')^2$,

or $\qquad x'=V'\,.\,t'+\tfrac{1}{2}\,.\,a'\,.\,t'^2$.

Or, again, it may—on occasion—be used as a method of determining what kind of form the expression of a particular (unknown) relation must take. *E.g.*, for the Simple Pendulum, the relation between length, periodic time, and acceleration of gravity, must conform to $\alpha=\lambda^{+1}\,.\,\tau^{-2}=\lambda/\tau^2$: giving indication of a relation of the type

$$g=n\,.\,l/t^2,\ \text{or}\ t=\sqrt{(n\,.\,l/g)},$$

where n is some number.†

* This relation is exhibited in the use of common terms (*second, minute*) for both *angles* and *times*, and in the whole technique of Spherical Astronomy. Proportionality of *time* to *length* in uniform *straight* motion is, by comparison, clearly not of general practical utility ; the moving particle does not remain " at hand ".

† The importance of these applications—of sub-section (ii)—of " Dimensions ", by comparison with their straightforward use (as in (i)) in determination of change of derived units, tends to be exaggerated.

9. *Measurement in general.*

The measurement of the Quantities of Physics—other than those of the basic kinds, *Length*, *Time*, ... , is, as indicated in § 4 above, not direct, but, progressively, by means of functional relations of the fundamental theoretical structure (§ 6, above) : these relations (of Joint Proportions or Multiplicate Proportion) being axiomatic propositions of the theory, stated either as " Definitions " (*e.g.* of *Velocity*, *Acceleration*, *Momentum*) or as " Laws " (*e.g.* of Motion, for *Mass* and *Force*).

And Addition-and-Subtraction, of quantities of the same kind, is achieved by addition-and-subtraction of their measures—as so determined. (See § 3, above.*)

10. *Notation and Terminology.*

Interpreted in terms of §§ 5, 6, above, the sign $\propto$—together with some such comparison sign as $\backsim\!\!\propto$ † (for inverse proportion)—has a precise functional significance in each particular case of a Physical relationship to which it is applicable. For example,

(1) *For the Rectangle*, if X, V denote the two *lengths* of its sides, and P its *area*, the proposition

$$P \propto X,\ V, \text{ jointly,}$$

expresses precisely the functional relation between the three quantities, *in that context* (when the rectangle is given, they are given ; and when it varies, they vary, *in that way*, with it).

(2) *For the representative* (*finite*) ‡ *Motion*, if X, V denote, respectively, the *length* and the *time*, and P the *speed*,‡ the functional relation is, similarly, expressible precisely by

$$\begin{aligned} P &\propto X \\ &\backsim\!\!\propto V, \text{ jointly ;} \end{aligned}$$

and (3) the most general type of functional relation of joint multiplicate proportion (§ 6 above) may be expressed precisely by

$$\begin{aligned} P &\propto X^h,\ Y^k, \ldots \\ &\backsim\!\!\propto V^l,\ W^m, \ldots, \text{ jointly.} \end{aligned}$$

* Use of the theoretical system of units is so general that the terms *length, time, mass, velocity, acceleration, force,* ...... , are commonly used to mean the measures (all, of course, " numbers ") of quantities properly so named, in terms of the units of the theory. The loose type of phrase exemplified by " length in feet ", " force in dynes ", arises out of that kind of usage.

† This is, in fact, a most useful symbol. Both symbols are of the " symmetric " type, like the sign of equality (and *unlike* the signs of inequality), and are, from that point of view, not quite well " designed ".

‡ " Speed " is defined, primarily, for a *finite* motion. Speed-at-an-instant, the important limiting case, is of course secondary to that. To call the former " the average speed " is, therefore, something like " putting the cart before the horse". The same is true of other similar cases, such as that of *curvature*.

Actually, the more compact forms

(1) $P = X . V$; (2) $P = X/V$; (3) $P = (X^h . Y^k \ldots)/(V^l . W^m \ldots)$

are in common use, for this purpose; but these forms, in fact, obviously derive from the *mathematical* relations of §§ 5, 6, 7 (in terms of ratios, *i.e.* numbers), which are consequences of the more fundamental functional relations of proportion between the physical quantities themselves.* The " expressions " (" products ", etc.) in them have no independent meaning †; they are simply a convenient concise way of expressing what are, in fact, relations of joint proportion; and their usefulness is largely bound up with the convenient practice of eliding (or ignoring) the distinction between physical quantities and their measures *in terms of the theoretical system of units.* (But *physical quantities are not numbers* ‡; and the said " convenient practice " has many serious consequences.§)

So, again, the notation L/T, L/T^2, $\mathsf{M} . \mathsf{L}/\mathsf{T}$, ... (or $\mathsf{L} . \mathsf{T}^{-1}$, etc.), is used for the " derived units " of Speed, Acceleration, Momentum, ...; and, in the general case, $\mathsf{L}^a . \mathsf{T}^b$ ....—where a, b, ... are the " dimensional numbers " (as in § 8, above). And L, T, ... and the derived units are, in fact, implicit in all such conventional specifications as " the length l ", " the time t ", " the velocity v ", ..., where l, t, v are actually *numbers* (viz. the *measures* of the several quantities—*length, time, speed,* ......—in terms of the respective units of the theoretical system).‖ The extent to which they should be made explicit is a matter for careful consideration.

D. K. Picken.

* It is not, of course, quite so straightforward as this: seeing that " multiplicate proportion " is a kind of corollary to joint proportion, arrived at through the mathematical expression (see § 6, *supra*). It should be noted that the basic case of multiplicate proportion is the relation of the " derived unit " of *area* to the unit of *length* (*i.e.* the relation of the *area* of a square to the *length* of its side).

† When we say that the *area* of a rectangle is the " product " of the two *lengths* of its sides, we are in fact making justifiable generalisation of definition of the term " product ". And similarly in the other cases in question. But the significance of this is obscured by the practice (referred to below) of ignoring the distinction between physical quantities and their measures.

‡ There are distinguished authorities who take the opposite view. But it is of the essence of the treatment here given that that view cannot finally be maintained. It appears to arise out of the fact that the theoretical system of units is not given its due place of absolute and essential importance in Physical theory; and to consequent inadequate realisation of the significance of the above-mentioned " convenient practice ".

§ For instance in elementary presentation of the fundamentals of Relativity, where the ordinary canons of measurement may not be assumed.

‖ The fact that this system of units is not merely (as is often stated) " the most convenient ", but is an essential of Physical theory, cannot be too strongly emphasised.

GLEANINGS FAR AND NEAR.

1397. The membership of the association has remained constant, and is in fact (at 4,630) two more than last year.—*Times Literary Supplement,* February 17, 1940. [Per Mr. A. P. Rollett.]

CORRESPONDENCE.

EXAMINATION QUESTIONS.

To the Editor of the *Mathematical Gazette*.

SIR,—The inclusion in your now abolished Pillory of questions which caused hardship to scholarship candidates on account of misprints or mistakes and of questions which exercised an injurious influence on Sixth-form teaching was obviously beneficial. Might it not be even more valuable to award stars to questions at the other end of the scale?

Scholarship papers must contain a large number of bread-and-butter questions, but should also contain some which test flair, insight and originality; no doubt suitable questions of this type are not easy to devise.

May I submit for the first award of a star the following problem taken from the Trinity Group examination, December 9, 1941, for entrance scholarships at Cambridge?

> " Five points in a plane are given, no three of them lying on a straight line. Prove that at least one of the quadrangles determined by a set of four out of the five points is convex."

It is unlikely that any candidate will have tackled previously a question of precisely this type; it therefore forms an excellent test of the ability of a candidate to think for himself. The fact that there may be some doubt as to what fundamental assumptions can be made is irrelevant because presumably the credit given by the examiner will depend on how the candidate sets out his assumptions and then develops his argument.

Both useful and interesting results might be obtained if you would ask for other recommendations for the award of a star.

CLEMENT V. DURELL.

TERMINOLOGY.

To the Editor of the *Mathematical Gazette*.

SIR,—With reference to R. S. Williamson's request on p. 217 of the *Gazette* for October, 1941, my young pupils refer to the rhombus as the diamond-shaped figure, and I find that this is in keeping with the dictionary meaning of " diamond-shaped ".

Yours, etc.,

CATHERINE W. M. SHERRIFF.

1398. What strikes the French visitor is the quality of the instructors, the clarity of their explanations, the precision of their questions, their patience and indulgence. Also the empirical nature of their teaching. Little place is given to theory. I saw on a blackboard some drawings of machines, never a mathematical formula. These young Anglo-Saxons need to get a good grip on reality.—André Maurois, quoted in the *Times*, March 2, 1940. [Per Mr. G. A. Bull.]

MATHEMATICAL NOTES.

1585. *Ovals of constant diameter.*

On p. 115 of the *Gazette* for May, 1941, there is an example of an oval having the property that the distance between any pair of parallel tangents is constant. This is only one type of a large class of ovals with the property. There are some interesting general theorems which are doubtless well known, though I have not seen them stated. The proofs are all simple and need not be given.

An *oval* here means a closed curve without inflexions, and if the distance between any pair of parallel tangents is constant and equal to d the oval is called an oval of constant diameter d.

(1) If parallel tangents at P and Q touch a CD-oval, then PQ is normal both at P and at Q.

(2) Conversely, an oval for which every chord which is normal at one end is normal also at the other is a CD-oval and all these chords are equal. Also the centres of curvature at the ends of any diameter are at the same point.

(3) A CD-oval may be traced by the ends of a rod which moves so that it is always rotating about some point on the rod, if the length is suitably chosen.

(4) By taking arbitrary arcs of any curves (subject to obvious restrictions) and supposing a rod to roll on each in turn, we may describe a CD-oval by tracing the path of the ends of the rod.

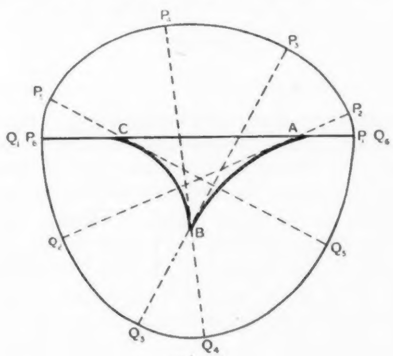


FIG. 1.

Let AB, BC be arcs of curves (Fig. 1) and P_1Q_1 the initial position of the rod. Rotate the rod about A to P_2Q_2, in which position it is tangent to the arc AB at A. Then roll the rod on the arc AB to P_3Q_3, rotate about B to P_4Q_4, roll along the arc BC and finally rotate about C. We must have as a condition for a closed oval

$$CQ_1 + \text{arc } BC = AP_1 + \text{arc } AB,$$

but by going twice round the curves we may trace the whole, using one end only of the rod.

This oval includes three circular arcs. We may eliminate these, if we wish, by taking the arcs AB, BC so as to touch CA at C and A and to touch one another at B.

(5) We can take an arbitrary arc AB of *any* convex curve, where the tangents at A, B are both perpendicular to AB as one "half" of a CD-oval (provided the radius of curvature at no point exceeds AB), and construct the other half by drawing normals at various points and cutting off constant lengths equal to AB on them. The easiest way is to draw circular arcs with centres on the curve and radii equal to AB. The envelope of these arcs gives the rest of the oval. In Fig. 2 the upper half is a semi-ellipse in which $a = 2b$.

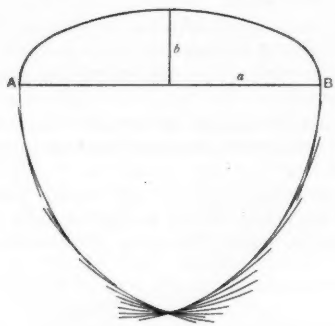


FIG. 2.

Such ovals will fit in any position into a square frame. A variation is to construct an oval which will fit in any position into an equilateral triangular frame, always touching the three sides. It is a neat geometrical exercise to prove, for such an oval, that:

(6) The normals at the three points of contact must meet in a point P, and the centres of curvature at the points are the vertices of an equilateral triangle whose circumcircle passes through P.

(7) The sum of the three radii of curvature is constant.

The simplest example is that of Fig. 3. Here $ABCD$ is a rhombus of side a with angles 60° and 120°. The oval consists of four circular arcs with centres A, B, C, D and radii r or $r + a$. This fits into an equilateral triangle of side $(6r + 2a)/\sqrt{3}$.

But there are ovals which defy both the square and the equilateral triangle tests. The envelope of the line

$$x \sin \psi - y \cos \psi = a - \tfrac{1}{24} b \cos 5\psi \qquad (a \geqslant b)$$

is an oval for which $\rho = a + b \cos 5\psi$ at the point "ψ", and its normals at ψ, $\psi \pm \frac{2}{3}\pi$, meet in the point $(\frac{5}{24} b \sin 6\psi, -\frac{5}{24} b \cos 6\psi)$.

This oval will fit an equilateral triangle of side $2\sqrt{3}a$, or a square of side $2a$ (or, clearly, a regular hexagon).

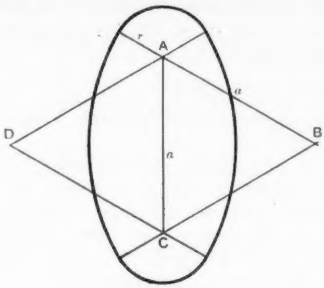


FIG. 3.

The problem of the oval to fit in all positions into a non-equilateral triangle is more exciting. Let a moving triangle ABC, whose angles are fixed, be drawn so as to circumscribe an oval. Then if the normals at the three points of contact are always concurrent, it can be proved that the sides of the triangle are constant. This leads to the condition that $a\rho_1 + b\rho_2 + c\rho_3 = \text{constant}$, where ρ_1, ρ_2, ρ_3 are the radii of curvature at the points of contact. It has been pointed out to me by Dr. E. D. Tagg that the oval $\rho = \lambda + \mu \cos (n+1)\psi$ has the required property, provided the angles of the triangle are of the form $\pi(n - 2m_r)/n$, $r = 1, 2, 3$, with integral m_r.

For example, $\rho = \lambda + \mu \cos 9\psi$ will fit an isosceles right-angled triangle.

G. W. BREWSTER.

1586. *Notes on Conics.* 7 : *Bifocal Distances.*

The most emancipating discovery in the whole of metrical pure geometry was that of the generality imparted by the use of algebraic instead of arithmetical measurements. Until it was possible to write, of any three collinear points, $AB + BC = AC$, instead of *If B is between A and C then $AB + BC = AC$, if C is between A and B then $AB - BC = AC$, and if A is between B and C then $BC - AB = AC$*, all but the simplest theorems were made sterile by the weight of their own multitudes of cases. The utilisation of algebraical measurements involves the fundamental principle that unless two steps are parallel, it is only the squares of their lengths that can be fruitfully compared, not the lengths themselves. We may replace algebraical measurements by arithmetical at any stage, enumerating the cases that may arise, but a theorem in which arithmetical lengths occur can not return into a general demonstration.

In the geometry of conics it is on the bifocal properties of conics that these considerations have their bearing. The relations

$$r + r' = 2a,\ r' - r = 2a,\ r - r' = 2a,$$

are vivid and important, but for inference we must go behind them

to one geometrical theorem or to a single algebraic relation unless all that follows is to seem to be a more or less artificial synthesis of distinct theorems. The equivalent theorem is familiar enough :

If H is the image of S in the tangent at a point P, then S', P, H are collinear and $|S'H|=2a$.

Or in a form purely geometrical,

The circle round P which passes through S touches the circle round S' whose radius is $2a$.

This theorem is, of course, established directly as a general theorem, not separately in different cases, and since $|HP|=r$, $|S'P|=r'$, we have $r+r'=2a$ if S and S' are on the same side of the tangent, that is, if the curve is an ellipse, since then P is between H and S', but $r\sim r'=2a$ if S and S' are on opposite sides of the tangent, that is, if the curve is a hyperbola.

As a first example, take a deduction which is in essence commonplace. If SY, $S'Y'$ are the perpendiculars from S, S' on the tangent, then

$$S'Y'/SY=S'Y'/YH=S'P/PH=(S'H-PH)/PH\ ;$$

also $SY\,.\,S'Y'=a^2(1-e^2)$. Hence, if p is the perpendicular length $|SY|$,

$$\frac{a^2(1-e^2)}{p^2}=\frac{S'H}{PH}-1.$$

Thus far the argument is general, and now we discriminate : the factor $1-e^2$ is to be written $-(e^2-1)$ if the curve is a hyperbola, and $S'H/PH$ is $2a/r$ or $-2a/r$ according as S' and P are on the same side or on opposite sides of H.

For an ellipse,

$$\frac{a^2(1-e^2)}{p^2}=\frac{2a}{r}-1,$$

for the branch of a hyperbola nearer to S,

$$\frac{a^2(e^2-1)}{p^2}=1+\frac{2a}{r},$$

and for the branch of a hyperbola further from S,

$$\frac{a^2(e^2-1)}{p^2}=1-\frac{2a}{r}.$$

For a second example, we have

$$2CP^2=S'P^2+SP^2-2CS^2=S'P^2+PH^2-2CS^2,$$

and therefore

$$CP^2+S'P\,.\,PH=\tfrac{1}{2}(S'H^2-2CS^2)=a^2(2-e^2).$$

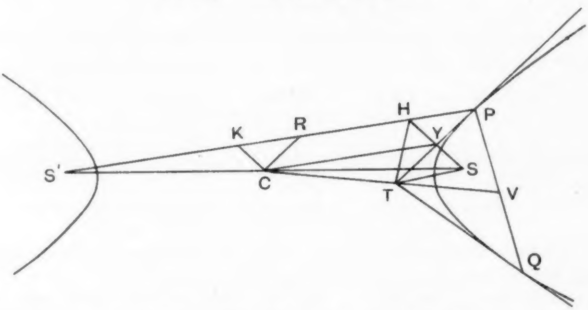
Since $a^2(2-e^2)$ is the orthoptic measure of the conic,

The product $S'P\,.\,PH$ is the radial measure along the diameter conjugate to CP.

If the conic is an ellipse, the conjugate radial measure is positive and the result can be written $rr' = CD^2$; if the conic is a hyperbola, $S'P \,.\, PH$ and the conjugate radial measure δ are alike negative, and the result is best expressed in the form that the conjugate radial measure is $-rr'$. But the proof of the equality precedes the recognition that the interpretation varies.

A relation between products of pairs of lengths may be a disguised relation between areas. Let the lines through C parallel to and perpendicular to the tangent at P cut $S'P$ in R and K; since CY is parallel to $S'P$, $PR = HK = \frac{1}{2}HS'$, and therefore if T is any point on the tangent at P,

$$\triangle TPC = \triangle TPR = \tfrac{1}{2}\triangle THS'.$$



If then TQ is the second tangent from T and V in the mid-point of PQ, we have

$$\triangle TPQ = 2\triangle TPV = (TV/TC)\triangle THS'.$$

The angles PTQ and HTS' are not necessarily equal, but the crosses (TP, TQ) and (TH, TS') are congruent, and therefore the arithmetical ratios

$$|\triangle TPQ|/|\triangle THS'|, \quad |TP|\,|TQ|/|TH|\,|TS'|$$

are equal, and replacing $|TH|$ by $|TS|$ we have

$$|TP|\,|TQ| = |TV/TC|\,|TS|\,|TS'|,$$

or in a striking form:

If TP, TQ are the tangents from a fixed point T to a variable member of a family of confocal conics, and V is the mid-point of PQ, the ratio $|TP|\,|TQ|/|TV|$ is constant.

Again the modification in the case of an ellipse is simple. Writing

$$\frac{TC}{TV} = 1 + \frac{VC}{TV} = 1 + \frac{TC \,.\, VC}{TC \,.\, TV},$$

we recognise the numerator $TC \,.\, VC$ as the radial measure along CT and the denominator $TC \,.\, TV$ as the power of T along the same

line. The ratio of one of these to the other is equal to that of the radial measure in any direction to the power of T in the same direction; in particular, if diameters parallel to TP, TQ cut the ellipse in E, F,

$$\frac{VC}{TV}=\frac{CE^2}{TP^2}=\frac{CF^2}{TQ^2}=\frac{|CE|\,|CF|}{|TP|\,|TQ|}=\frac{|\triangle CEF|}{|\triangle TPQ|}.$$

Thus $|\triangle THS'|=|\triangle TPQ|+|\triangle CEF|$ and

$$|TP|\,|TQ|+|CE|\,|CF|=|TS|\,|TS'|.$$

For the hyperbola, if E', F' are points in which the diameters parallel to the tangents cut the conjugate curve,

$$|\triangle CE'F'|/|\triangle TPQ|=CV/TV,$$

and the arithmetical result depends on whether this ratio is greater than or less than unity. If P and Q are on the same branch,

$$|CT|=|CV|-|TV|,\quad |\triangle THS'|=|\triangle CE'F'|-|\triangle TPQ|,$$
$$|CE'|\,|CF'|-|TP|\,|TQ|=|TS|\,|TS'|;$$

if P and Q are on different branches, $|TC|=|TV|-|CV|$, implying $|\triangle THS'|=|\triangle TPQ|-|\triangle CE'F'|$,

$$|TP|\,|TQ|-|CE'|\,|CF'|=|TS|\,|TS'|.$$

It is salutary to observe that we shall not necessarily obtain a true result for a hyperbola by " writing $CE^2=-CE'^2$, $CF^2=-CF'^2$ in the result for an ellipse "; the horrible process does not even " work ".

E. H. N.

1587. *An extension of Newton's formula for approximating to the roots of equations.*

Investigations connected with the calculation of zeros of certain transcendental functions have led to the development of direct methods to replace those of numerical successive approximation. Essentially the method consists in developing the zero of $f(x)$ to which $x=a$ is a first approximation as a power series of a (small) quantity proportional to $f(a)$, with coefficients which are functions of a; the properties of the particular function are used to determine these coefficients. The well-known Newton approximation,

$$x=a-f(a)/f'(a)$$

can be regarded as the first two terms of such a series, and the purpose of this note is to indicate how such a series may be obtained in the general case, to give a few more of its terms, and to illustrate their use.

If a is an approximation to a root of the equation

$$f(x)=0, \quad\text{.................................(1)}$$

and if $a+h$ is the exact value, then

$$f(a+h)=0=f(a)+hf'(a)+h^2f''(a)/2!+h^3f'''(a)/3!+\ldots\ldots\text{(2)}$$

Divide through by $f'(a)$, replace $f(a)/f'(a)$ by θ, and denote (for $n>1$) $f^{(n)}(a)/f'(a)$ by F_n, and obtain

$$-\theta = h + F_2 h^2/2! + F_3 h^3/3! + \dots \quad \text{......................(3)}$$

Reverting this series, we deduce that

$$h = -\theta - A_2\theta^2 - A_3\theta^3 - \dots \quad \text{......................(4)}$$

The coefficients in (4) may be found *ad hoc—formal* successive approximation—or by use of the formulae given by van Orstrand [*Phil Mag.* (6) xix, 366, (1910)]. We find

$$A_2 = F_2/2, \quad A_3 = (3F_2^2 - F_3)/6,$$
$$A_4 = (15F_2^3 - 10F_2F_3 + F_4)/24,$$
$$A_5 = (105F_2^4 - 105F_2^2F_3 + 10F_3^2 + 15F_2F_4 - F_5)/120,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

As a first example let us use

$$f(x) = x^2 + 2x - 23 = 0. \quad \text{..........................(5)}$$

The positive root is $\sqrt{24} - 1 = 3{\cdot}89897\,94856 \dots$. So, using 4 as a first approximation, we have

$$a = 4,\ f(a) = 1,\ f'(a) = 10,\ f''(a) = 2,\ f^{(n)}(a) = 0 \quad (n>2),$$
$$\theta = 0{\cdot}1,\ F_2 = 0{\cdot}2,\ F_n = 0 \quad (n>2),$$

so that

$$\begin{aligned} h &= -\{0{\cdot}1 + (0{\cdot}2)(0{\cdot}1)^2/2 + (0{\cdot}2)^2(0{\cdot}1)^3/2 \\ &\qquad + 5(0{\cdot}2)^3(0{\cdot}1)^4/8 + 7(0{\cdot}2)^4(0{\cdot}1)^5/8 + \dots\} \\ &= -0{\cdot}10102\,0514 \dots, \end{aligned}$$

giving as the root

$$x = 3{\cdot}89897\,9486 \dots .$$

As a second example we take the equation

$$x - \cot x = 0$$

to the smallest positive root of which, $0{\cdot}86033\,36 \dots$,

$$\pi/4 = 0{\cdot}78539\,8 \dots$$

is a not so very close approximation. Here $\theta = -0{\cdot}07153 \dots$, and the coefficients and terms of the series are as follows :

$$\begin{array}{llr} & a = & +0{\cdot}78539\,82 \\ & -\theta = & +0{\cdot}07153\,29 \\ A_2 = -\frac{2}{3} & -A_2\theta^2 = + & 341\,14 \\ A_3 = 0 & & \\ A_4 = +\frac{10}{27} & -A_4\theta^4 = - & 97 \\ \hline & & \underline{0{\cdot}86033\,38} \end{array}$$

These examples may suffice to show how the series enables a direct computation of a root to be effected. Of course, the advan-

tages of a close first approximation are reaped by this method, and this is the more true since the series so developed are, in general, not convergent but asymptotic. W. G. B.

P.S. Surely someone has done this before—but if so, who, when, and where? W. G. B.

1588. *When mass varies.*

Most of us, I believe, starting with the equation of motion in the form

$$F = m\ddot{s} = m\dot{v}, \quad \text{.............................(1)}$$

will emphasise the generality of the two integrals

$$\int F\,dt = [mv], \quad \text{.............................(2)}$$

$$\int F\,ds = [\tfrac{1}{2}mv^2], \quad \text{...........................(3)}$$

which express respectively the principles of impulse-momentum and work-energy—but imply *constant mass.*

For some time I have insisted that force is *rate of change of momentum*, and so upon the use of the more general equation of motion,

$$F = \frac{d}{dt}(mv), \quad \text{...............................(4)}$$

from which (2) still follows, but only recently have I asked myself what corresponds to (3) in this case. Evidently not an *energy* integral, since variation of mass involves " shock " and dissipation of (mechanical) energy.

Now (3) follows from (1) upon multiplication by v and integration. To obtain the corresponding result from (4) we multiply by mv, and obtain

$$Fmv = (mv)\frac{d}{dt}(mv),$$

$$\int Fm\,ds = [\tfrac{1}{2}(mv)^2]. \quad \text{.............................(5)}$$

The result still involves only the product mv—momentum. *No* new quantity analogous to kinetic energy makes its appearance.

In the common instance, the fall of a chain coiled on a floor through a hole, we have, if x is the length hanging vertical at any time,

$$gx = \frac{d}{dt}(x\dot{x}) = \dot{x}^2 + x\frac{d}{dx}(\tfrac{1}{2}\dot{x}^2). \quad \text{....................(6)}$$

An integrating factor is x, so that

$$\tfrac{1}{3}gx^3 = \tfrac{1}{2}x^2\dot{x}^2 + C, \quad \text{.............................(7)}$$

which is the equivalent of (5) in this case.

The moral is the fundamental importance of the (mechanical) concept of momentum—but the surprise is that the *degenerate* case (3) introduces a *new* quantity, energy. W. G. B.

1589. *On a neglected equation in dynamics.*

It is well known that the solution of certain types of dynamical problem is much simplified if when forming the equation of moments we take as origin of moments a point moving in any arbitrary manner. Thus if we wish to avoid the introduction of some unknown force into our equations we take moments about its point of application.

For a rigid body moving in two dimensions, the equation of moments is very conveniently given in the form

$$I_G\omega + Mpf = N,$$

where M is the mass of the body, I_G the moment of inertia about the centroid G, $\mathbf{f}$ the acceleration of G and f its magnitude, p the perpendicular from O on the vector $\mathbf{f}$, and N the moment about O of the external forces.

For proof, we consider an element of mass m, and take moments of its equation of motion about O; summing as usual, and noting that the contributions from the internal forces cancel, we have

$$\Sigma m\mathbf{r}_\wedge(\ddot{\boldsymbol{\rho}} + \mathbf{f}) = \Sigma\mathbf{r}_\wedge\mathbf{F}, \qquad \text{(i)}$$

where $\mathbf{F}$ is the external force on m. But

$$\ddot{\boldsymbol{\rho}} = -\rho\dot{\theta}^2\boldsymbol{\rho}_1 + \rho\ddot{\theta}\boldsymbol{\rho}_2,$$

where $\boldsymbol{\rho}_1$ is a unit vector in $\boldsymbol{\rho}$, $\boldsymbol{\rho}_2$ a unit vector perpendicular to $\boldsymbol{\rho}$ in the direction of the rotation of the body, and ρ the magnitude of the vector $\boldsymbol{\rho}$.

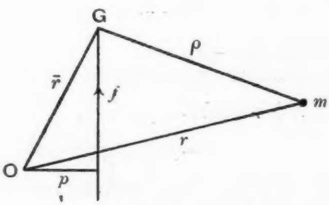


FIG. 1.

Substituting and using the fact that $\Sigma m\boldsymbol{\rho} = 0$, since G is the centroid, the left-hand side of (i) becomes

$$\begin{aligned}&\Sigma m(\rho\boldsymbol{\rho}_1 + \bar{\mathbf{r}})_\wedge(-\rho\omega^2\boldsymbol{\rho}_1 + \rho\dot{\omega}\boldsymbol{\rho}_2 + \mathbf{f})\\ &= \Sigma m\rho^2\dot{\omega}\boldsymbol{\rho}_{1\wedge}\boldsymbol{\rho}_2 + \Sigma m\bar{\mathbf{r}}_\wedge\mathbf{f}\\ &= I_G\dot{\omega}\mathbf{n} + M\bar{\mathbf{r}}_\wedge\mathbf{f},\end{aligned}$$

where $\mathbf{n}$ is the unit vector normal to the plane.

Taking the magnitudes of the vectors, we obtain

$$I_G\dot{\omega} + Mpf = N.$$

Though this equation is very general and is easily applicable, it is not referred to in many textbooks; even when mentioned, it is hardly emphasised and is seldom used in illustrative examples given in the books. Whereas the formula for moving axes in three dimensions is very frequently used, this corresponding two-dimensional formula is neglected. To indicate its advantages, the equation is applied below to two examples taken from Lamb's *Dynamics*, Ch. IX, 63.

Ex. 2. A solid of revolution, with axis horizontal, rolls down a plane of inclination α.

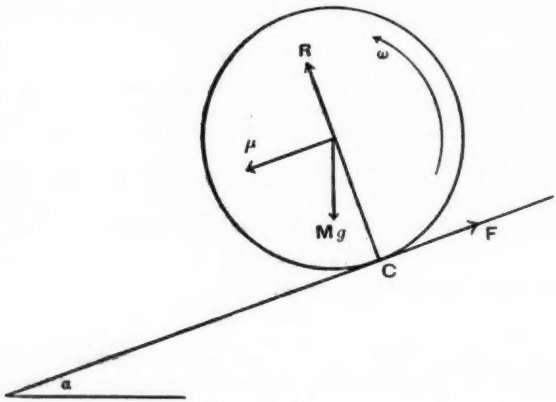


FIG. 2.

The rolling condition is $u - a\omega = 0$.

Taking moments about the point of contact, C, we have

$$Mk^2\dot{\omega} + Ma\dot{u} = Mga \sin \alpha,$$

or

$$\dot{\omega} = ga \sin \alpha/(k^2 + a^2).$$

Ex. 3. A cylinder of mass M and radius a rolls in a fixed cylinder of radius b, the sections being circular and the axes of symmetry parallel and horizontal.

The rolling condition is $a\omega = (b-a)\dot{\theta}$.

Taking moments about the point of contact

$$Mk^2\dot{\omega} + Ma(b-a)\ddot{\theta} = -Mga \sin \theta.$$

Hence

$$(k^2 + a^2)(b-a)\ddot{\theta} = -ga^2 \sin \theta.$$

It seems therefore not out of place to urge teachers and writers of textbooks to make more frequent use of this equation.

C. JAYARATNAM ELIEZER.

1590. *The volume of the pyramid.*

The formula of the volume (V) of a pyramid is found by means either of the calculus from the definite integral

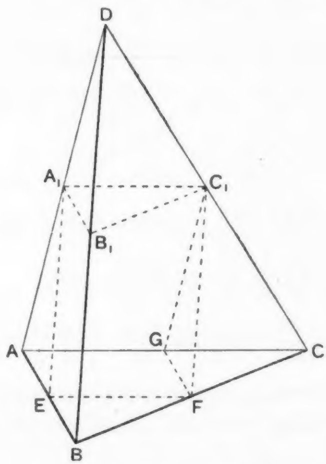
$V=\int_0^h A\,dx$, where A is the section parallel to the base, and h the height of the pyramid ;

or, $V=\int_0^h \frac{S}{h^2}x^2\,dx$, where S is the base of the pyramid,

$$=\frac{S}{h^2}\int_0^h x^2dx=\tfrac{1}{3}Sh\ ;$$

or without using the calculus by proving that a prism can be divided into three pyramids of equal volume.

The following proof is based solely upon the properties and the formula of the *sum of a convergent geometrical progression.*



Let any pyramid * having the base S and the height h be divided by three sections as follows :

1st section. A plane parallel to the base, distant $h/2$, cuts off the small pyramid $A_1B_1C_1D$.

2nd section. Another plane parallel to the side-face ABB_1A_1 through C_1 cuts off from the frustum of the pyramid a second pyramid $CFGC_1$ which is congruent to the pyramid $A_1B_1C_1D$.

* The figure shows a triangular pyramid for the sake of simplicity, but the reasoning is general and the proof applies to *any* pyramid.

H 2

3rd section. A third plane through the edge A_1C_1 parallel to the edge BB_1 cuts up the remaining body into the prism $EBFA_1B_1C_1$ and the wedge $AEFGC_1A_1$.

Let pyramid $ABCD$ be P,
wedge $AEFGC_1A_1$ be W,
prism $EBFA_1B_1C_1$ be Pr,

then pyramid $A_1B_1C_1D$ = pyramid $CFGC_1$.

If pyramid $CFGC_1$ be P_1, we have

$$P = W + Pr + 2P_1; \qquad \text{(1)}$$

$$W = \frac{S}{2}\cdot\frac{h}{2}\cdot\frac{1}{2} = \frac{Sh}{8}, \quad S \text{ being the base of the pyramid};$$

$$P_r = \frac{S}{4}\cdot\frac{h}{2} \quad = \frac{Sh}{8}.$$

Hence
$$P = \frac{Sh}{8} + \frac{Sh}{8} + 2P_1 = \frac{Sh}{4} + 2P_1. \qquad \text{(2)}$$

P_1 can be broken up again in the same way as P; that is,

$$P_1 = \frac{S_1 . h_1}{4} + 2P_2, \quad \text{where } S_1 = \frac{S}{4},\ h_1 = \frac{h}{2};$$

$$P_1 = \frac{S}{4}\cdot\frac{h}{2}\cdot\frac{1}{4} + 2P_2 = \frac{Sh}{32} + 2P_2;$$

$$P_2 = \frac{S_2h_2}{4} + 2P_3, \quad \text{where } S_2 = \frac{S_1}{4},\ h_2 = \frac{h_1}{2};$$

$$P_2 = \frac{S_1}{4}\cdot\frac{h_1}{2}\cdot\frac{1}{4} + 2P_3 = \frac{S_1h_1}{32} + 2P_3;$$

$$P_2 = \frac{Sh}{256} + 2P_3, \text{ as } \frac{S_1h_1}{32} = \frac{S}{4}\cdot\frac{h}{2}\cdot\frac{1}{32} = \frac{Sh}{256}.$$

P_3 and all the following pyramids could be divided in the same way.

If the corresponding values of $P_1, P_2, P_3, \ldots$ are substituted in (2), we have

$$P = \frac{Sh}{4} + 2\left(\frac{Sh}{32} + 2P_2\right)$$

$$= \frac{Sh}{4} + \frac{Sh}{16} + 2P_2,$$

$$P = \frac{Sh}{4} + \frac{Sh}{16} + 4\left(\frac{Sh}{256} + 2P_3\right),$$

$$P = \frac{Sh}{4} + \frac{Sh}{16} + \frac{Sh}{64} + 8P_3. \qquad \text{(3)}$$

Equation (3) provides enough terms to find the law of formation of

the successive terms—by means of the principle of Mathematical Induction ; thus the general form of equation (3) is

$$P=\frac{Sh}{4}+\frac{Sh}{4^2}+\frac{Sh}{4^3}+\ldots+\frac{Sh}{4^n}+2^n \,.\, P_n. \quad \ldots\ldots\ldots\ldots\ldots(4)$$

The right-hand side of this equation requires the summation to infinity of a geometrical progression ; we have only to show that the additive term

$$2^n \,.\, P_n$$

tends to zero as its limit as n is indefinitely increased.

After cutting off from the pyramid $ABCD$ the four congruent pyramids $A_1B_1C_1D$, $CFGC_1$, $AEGA_1$, and $EBFB_1$, we have a prismatoid or double pyramid—which in case of $ABCD$ being a tetrahedron becomes an octahedron.

Thus we have

$$2P_1<\frac{P}{2}\,;\quad 2P_2<\frac{P_1}{2}\,;\quad 2P_3<\frac{P_2}{2}\,;\ldots;\quad 2P_n<\frac{P_{n-1}}{2}\,;$$

Thus $$2P_1<\frac{P}{2}\,;\quad 4P_2<\frac{P}{4}\,;\quad 8P_3<\frac{P}{8}\,;\ \ldots;\ 2^nP_n<\frac{P}{2^n}.$$

The sum to infinity of the series

$$\frac{Sh}{4}+\frac{Sh}{4^2}+\frac{Sh}{4^3}+\ldots+\frac{Sh}{4^n}$$

is obtained by means of the formula

$$\Sigma_\infty=\frac{a}{1-r},\quad \text{where } a=\frac{Sh}{4},\ r=\frac{1}{4}.$$

Thus we have $$\Sigma_\infty=\frac{Sh}{4(1-\frac{1}{4})}=\frac{Sh}{3}.$$

That shows that the volume of a pyramid is $\frac{1}{3}$ of the product of base and height. REV. JOSEF BECKER, S.V.D.

1591. *Solution of quadratics with real roots.*

If the numerical solution of only one or two quadratics is required, the method used is of little importance, though the standard formula is certainly the least convenient.

If the solution of a batch of twenty or thirty quadratics is required, efficient methods are essential. Horner's method is reliable and can be done entirely on a calculating machine, but is rather slow. Methods using continued fractions have the same defect.

The following iterative formula was found to be rapid and foolproof. It is unlikely to be new, but I have not come across it before.

" If x_n is an approximation, correct to n figures, to a root of $x^2+bx+c=0$, then the approximation

$$x_{n+1}=\tfrac{1}{2}(x_n^2-c)/(x_n+\tfrac{1}{2}b)$$

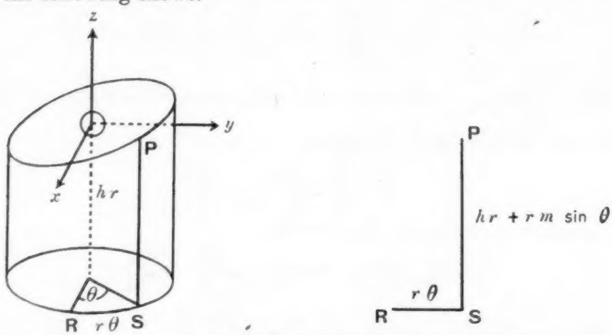
is correct to $2n$ figures."

When $b=0$, this reduces to Newton's iterative approximation for square roots.

A good flying start makes the method very rapid. For this a nomogram for the solution of quadratics is useful; for example, we can use that given by Professor Bickley in the *Gazette*, XXV, No. 264. *Barlow's Tables* are, of course, assumed to be handy. R. A. FAIRTHORNE.

1592. *Section of a circular cylinder.*

Note 1539 (October, 1941) reminds me of an unsuspected relationship which came to light when dealing with plane sections of the common geometrical solids. To illustrate the results, I made cardboard models, and was led to consider the shape of the plane figure from which could be developed a cylinder cut by a plane not at right angles to its axis. It proved to be bounded by a sine curve, as the following shows.



Let the equations of the cylinder and cutting plane be

$$x^2+y^2=r^2, \qquad z=my,$$

so that the coordinates of a point on the curve of intersection may be expressed as $(r\cos\theta,\ r\sin\theta,\ mr\sin\theta)$. It is clear, then, that the cylindrical surface between the planes $z=my$ and $z=-hr$ can be developed from a plane figure bounded by two rectangular coordinate axes, the curve $(h+m\sin\theta)$ and the ordinate $\theta=2\pi$.

D. T. ROBERTS.

1593. *On a problem in elementary geometry.*

The main results obtained by Messrs. Maunsell and Pedoe under this heading in the July *Gazette* seem to be obtained more naturally as follows.

The starting point is the sloping-hillside problem in which the relation between an angle and its projection is usually met for the first (and by many students for the only) time.

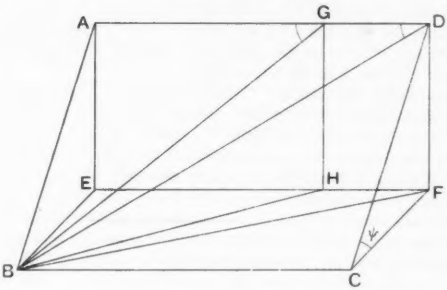
If $\angle GBA=\theta_1$, projecting into $\angle EBH=\phi_1$, and the slope of the plane is ψ, then $AG=AB\tan\theta_1$ and $EH=AB\cos\psi\tan\phi_1$. Thus

$$\tan\phi_1=\tan\theta_1\sec\psi.$$

So also if $\angle DBA = \theta_2$, projecting into ϕ_2, $\tan \phi_2 = \tan \theta_2 \sec \psi$. Hence the condition that $\theta_2 - \theta_1$ should be greater than $\phi_2 - \phi_1$, its projection either on $EBCF$ or on a parallel plane through AD, is

$$\tan(\theta_2 - \theta_1) > \tan(\phi_2 - \phi_1),$$

i.e.
$$\frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1} > \frac{(\tan\theta_2 - \tan\theta_1)\sec\psi}{1 + \tan\theta_2 \tan\theta_1 \sec^2\psi}$$
$$= \frac{\tan\theta_2 - \tan\theta_1}{\cos\psi + \tan\theta_2 \tan\theta_1 \sec\psi}.$$



This condition is

$$\cos\psi + \tan\theta_2 \tan\theta_1 \sec\psi > 1 + \tan\theta_2 \tan\theta_1,$$

or
$$\tan\theta_2 \tan\theta_1 (\sec\psi - 1) > 1 - \cos\psi,$$

or
$$\tan\theta_2 \tan\theta_1 > \cos\psi.$$

If the angles AGB, ADB, the complements of θ_1, θ_2, are called λ_1, λ_2, the condition becomes $\cot\lambda_1 \cot\lambda_2 > \cos\psi$.

The theorem may be worded thus :

An angle α is projected at an angle ψ. If the arms of the angle α make acute angles λ_1, λ_2 with the axis of projection, then α will be greater than its projection if and only if $\cot\lambda_1 \cot\lambda_2 > \cos\psi$ *; it will be greater than its projection at* any *angle if* $\cot\lambda_1 \cot\lambda_2 > 1$.

This condition can readily be shown to be equivalent to that given in the July *Gazette*. If the bisector of α makes an acute angle γ with the axis of projection then λ_1, λ_2, are $\gamma \pm \frac{1}{2}\alpha$, and

$$\cot(\gamma + \tfrac{1}{2}\alpha)\cot(\gamma - \tfrac{1}{2}\alpha) = \frac{\cos\alpha + \cos 2\gamma}{\cos\alpha - \cos 2\gamma}.$$

$$\begin{cases} > 1 \text{ if } \gamma < \tfrac{1}{4}\pi, \\ > \cos\psi \text{ if } \cos\alpha \tan^2 \tfrac{1}{2}\psi + \cos 2\gamma > 0. \end{cases}$$

The condition $\gamma < \frac{1}{4}\pi$ given by Messrs. Maunsell and Pedoe is clearly the simplest form of the condition for all angles of projection, while $\cot\lambda_1 \cot\lambda_2 > \cos\psi$ seems the simplest form for any special angle.

C. O. TUCKEY.

REVIEWS.

A Short History of Science, to the Nineteenth Century. By CHARLES SINGER. Pp. xiv, 399. 8s. 6d. 1941. (Humphrey Milford, Oxford University Press)

This book is a pleasure to read. Dr. Singer has the knack of making us want more. His style is readable, his ideas are stimulating, and his point of view is human. It is difficult, for example, to refuse sympathy to an author who thus describes Hippocrates the physician: "Learned, observant, humane, with a profound reverence for the claims of his patients, but possessed of an over-mastering desire that his experience should benefit others; orderly and calm; anxious to record his knowledge for the use of his brother physicians and for the relief of suffering; grave, thoughtful and reticent; pure of mind and master of his passions; such is the image of the father of medicine as it appeared to his successors."

Dr. Singer begins by asking, "What is *science?*", and he rightly protests against the idea that it is merely a static body of knowledge. We should not speak of a "scientific" boxer. Rather is science an active process, a seeking, and the history of science is thus no side-line, but a wider, small-scale view of what is going on in every research laboratory. This process cannot ordinarily be viewed on the large scale, not at least by mere reading, for the accepted form of scientific writing (p. 250) unfortunately precludes any view of the mind of the writer or the (large-scale) history of his work.

Dr. Singer does not ask the complementary question, "What is *history?*", but we can perhaps guess what his answer would be. He says (p. 1) that because science is an age-long process, it "therefore behoves the historian of science to be very charitable, very forbearing, very humble, in his judgments and presentations of those who have gone before him". He also says (p. 227) that the man of science must choose his facts, as the poet chooses his words, "exercising his judgment to select those things which bear a certain relation to each other". The historian, too, must choose his facts, and his art consists in arranging them in an intelligible pattern, not so as to illustrate his own preconceived notions, but so as to exhibit their own natural relationships as perceived by his mind.

Judged by this standard, Dr. Singer is a true historian. Of course, none of us would make quite the same selection of facts. Mathematicians will perhaps be disappointed to find Pascal dismissed in three lines, d'Alembert mentioned only as a meteorologist, and no mention at all of Barrow, de Moivre, Monge, Lobachewski, Galton. More strange is the omission of Pasteur. We are warned in the preface that only physical and biological problems are to be considered, though this does not in fact mean that mathematics and philosophy are altogether excluded.

He describes (p. 189) how mathematics became, in the seventeenth century, the instrument of physical investigation, but he does not discuss the beginnings of quantitative work in the biological sciences. Indeed he is probably averse to any such idea, for he repudiates (p. 236) the suggestion that the aim of biology should be to represent biological phenomena in physical terms, while admitting that most of the biological advances of the seventeenth century were of such a nature. Mathematical treatment does not necessarily imply such an aim, and we might have hoped that the rise of statistical science would have been mentioned. Such a work as Galton's *Natural Inheritance* was surely of great significance for the future of biology.

To say that biology lags behind physics is admittedly unfair, for there is no parallel in physics to the vast field of qualitative and descriptive work

which the biological sciences offer; and to say the same thing, meaning that biology, *in its approach to the methods of physics*, lags behind physics, is to state the obvious. But this does not alter the fact that the introduction of quantitative methods is a landmark in the history of any science. Compare, for example, the progress of chemistry before and after the year 1750.

Interesting questions arise as to why such developments did not take place earlier. To what extent must one science wait on another? Discoveries made before their time are apt to be ignored. Why, for example, did Mendel's work attract no attention in the nineteenth century? Why was there such a long interval between the invention of the microscope and the discoveries made by Pasteur? And why was the synthetic geometry of the seventeenth century not developed until the nineteenth? Perhaps some future historian of science will answer these and the very many similar questions.

Dr. Singer recommends frequent reference to his "Table of Contents" as a guide to the somewhat intricate pattern formed by the history of scientific thought. He also helps us to find our way by giving page-references wherever he makes incidental mention of writers whose work he has described more fully elsewhere. This is a most commendable practice. There is, furthermore, an index of names, though not of subjects.

Some idea of the broad outlines of the pattern may be gained from the chapter headings. These are:

I. *The Rise of Mental Coherence.*
II. *The Great Adventure* (Unitary systems of thought: Athens).
III. *The Failure of Nerve* (Divorce of Science and Philosophy: Alexandria).
IV. *The Failure of Inspiration* (Science the handmaid of practice: Imperial Rome).
V. *The Failure of Knowledge* (The Middle Ages. Theology, Queen of the Sciences).
VI. *The Revival of Learning.*
VII. *The Insurgent Century.*
VIII. *The Mechanical World.*

In Chapters V and VI he brings out the very far-reaching fact that the only Greek texts handed on by the Arabs were those dealing with Philosophy, Medicine and Mathematics. These by themselves produced no renaissance. The humanists (p. 166) discovered the literary works of antiquity and were absorbed in them to the exclusion of all else, to the great detriment of future culture and education. The reconstruction of the Greek spirit and the rise of modern science became possible only when these two traditions were united in the persons of such men as Leonardo da Vinci, Vesalius, Galileo. It is tempting to add other names to this distinguished trio, but the list would soon become over-long. The conclusion would be even more striking, and, as Dr. Singer remarks, "not without bearing on our present state".

It is safe to say that those who take this view will enjoy reading Dr. Singer's very admirable book. It is perhaps worth adding that, at least as far as the first printing is concerned, their enjoyment will be the greater for its being printed on pre-war paper. E. H. L.

Lectures on Topology. Papers presented at the University of Michigan Conference, 1940, edited by R. L. Wilder and W. L. Ayres. Pp. vii, 316. \$3. 1941. (University of Michigan Press; Humphrey Milford)

This volume is a collection of lectures and papers read to a Topology Congress held in the University of Michigan in June and July, 1940. The congress was originally planned on an international basis, but it is not surprising, con-

sidering the date, that only mathematicians resident in the United States were able to attend. But it is good to know that it was possible to hold such a congress at all, and the remarkable diversity of topics covered is a tribute to the strength of the American schools of Topology. There are, of course, some gaps, but the range and importance of the topics discussed is very impressive.

A convention of mathematicians has to fulfil several purposes. What is perhaps its most important is never recorded and only shows itself indirectly later on. This is to provide an opportunity for discussion between specialists who otherwise have little opportunity of meeting, and while these are off the record of the conference they usually result in important advances in knowledge in subsequent months. Another important function of a congress is fulfilled by those experts who deliver lectures surveying the state of their chosen subjects, setting forth the results achieved and pointing out the difficulties still to be surmounted. These surveys are of the greatest value, for they serve as links to keep the whole congress together, and spread the knowledge of the experts amongst mathematicians whose primary interests are in different fields and so counter the disadvantages of excessive specialisation. There are also small sectional meetings to which papers on specialised topics are read. These have a more restricted sphere of influence, but serve a useful purpose in circulating ideas.

The more important of the shorter papers usually appear at some time or other in the usual scientific journals, and hence when the proceedings of the conference are published it is the general surveys which make the most interesting reading. In the present instance, most of these are masterly, and prove that the conference must have been a successful one. It is, of course, clear that not all of the lectures could have been delivered exactly in the form in which they are printed; no doubt in some cases only summaries of the printed versions could have been delivered by word of mouth. But the care with which the printed versions have been prepared leaves little doubt that one at least of the main functions of the congress was successfully fulfilled. Since the congress was devoted to Topology and did not cover any other branches of mathematics, the lectures are on the whole more specialised than one would expect at a general conference of mathematicians, but a mathematician who is not a topologist can obtain from these proceedings a very fair picture of the work which is being done by professional topologists.

The topics covered in the main lectures—there are twelve of them—vary from pure abstract Topology to applications to problems of great general interest. It is not possible to describe all of these lectures in this notice; indeed, it is not necessary to do so, for the reviewer's chief function is to tell readers that in this volume he will find a series of papers which will give him an excellent picture of the work which is being done by a number of the leading schools of topologists. It is perhaps invidious to select any for special mention, but certain of the lectures, such as, for instance, those by Lefschetz on the Abstract Theory of Complexes, by Eilenberg on Continuous Mappings, and by Cairns on Triangulated Manifolds, may prove of special interest to those mathematicians who seek to extend their knowledge of what topologists are doing. In Lefschetz's lecture a very lucid account of the modern abstract theory of complexes is given, and the reader will be struck by the tremendous increase in the power of the weapons employed compared with those used in what the lecturer calls "the old Topology" of his Colloquium Lectures, though the ultimate aim of the work is still as firmly kept in view as ever. Eilenberg's lecture gives a clear account of the efforts being made to organise homotopic theory, or the topology of deformation, into a systematic whole.

The lecture of Cairns will appeal particularly to those who are mainly interested in the less exotic types of space, and are interested in Topology mainly as an additional means of investigating constructs which arise in other branches of mathematics. Cairns' lecture is not in itself applied Topology, but it is concerned particularly with the topology of spaces which arise in less abstract branches of mathematics.

The lectures which have been mentioned have been selected somewhat arbitrarily, and there are others which will appeal just as much to the general mathematician who seeks to broaden his outlook, and indeed, all of the twelve deserve mention, and those actually referred to are not to be regarded as exceptional. Many readers may, for instance, find the first part of Whitney's paper on Differentiable Manifolds particularly attractive; it appears like a miniature in a collection of full-length portraits. It is, moreover, only fair to say that the shorter papers are all worth reading too.

The production of the book is excellent, and the volume can be recommended to all who wish to keep in touch with the course of mathematical research.

W. V. D. H.

A Mathematical Refresher. By A. Hooper. Pp. viii, 308. 9s. 6d. 1941. (Oliver & Boyd)

"This book is intended primarily for R.A.F. candidates who find their mathematics rusty through disuse or who, brought face to face with facts, have to admit that they have never properly grasped the subjects at school." So runs the opening sentence of the preface. On the wrapper a hope is expressed that the book "may also make an appeal to that much wider public who assert that they 'can't do maths.'. It may possibly help the man who has a mathematical inferiority complex foisted on to him by the type of teaching he has had to endure at school to discover that after all he '*can* do maths.'." "Learned mathematicians should they chance to light on the book" we read later in the preface "will shudder when they see, for instance, oranges mixed with Latitude and Longitude. But if the average man finds an orange helps him to grasp these things let him have an orange."

The reader who is gulled by all this into thinking that here at last is a book which will enable him to "master the beautiful simplicity and sequence of mathematical laws and processes" is due for a disappointment. The refreshment (oranges and all) is no more palatable than the ordinary run of school textbooks and costs very much more. If it were easier to digest the high cost might be excused, but the fare consists largely of rules, regulations and cautions on how to deal with the standard textbook problems. The exercises are grouped under chapter headings at the end of the book, and unlike other recent publications for the same market surprisingly few are of aviation interest. The text is written in a man-to-man style and much heavy type and italics are used. It is sometimes obscure; to take three examples:

"Thus, although 1 lb. weight = 1 lb. mass, the two things are not really the same."

"Multiply £17 8s. $9\frac{3}{4}$d. by 67.

1st method:

£	s.	d.
17	8	$9\frac{3}{4}$
		67
1168	10	$5\frac{1}{4} \leftarrow$

"N.B. In a converse you are given the opposite of the same theorem. A

converse is not *always* true, *e.g.* ' I love you ' does not necessarily mean ' you love me '."

The Geometry contains few formal proofs. (" They are not necessary, so why continue to inflict them? ") This is perhaps as it should be in a book of this sort, but why then include in the catalogue of geometrical facts many that are far too obvious and/or unuseful to be worth mentioning, unless some fun is to be got out of the game of assuming some of them to prove others. It is a pity that we are not given two sides and the angle opposite the larger side as a set of data more general than R.H.S. for determining a triangle.

The Algebra is sound, but it contains some very unrefreshing sections on such topics as factorising harder quadratic trinomials.

The book contains a little extra that some others haven't got, the last eight chapters being on: simultaneous and quadratic equations; twelve important facts about a circle; velocity and acceleration; gravity; relative velocity and the parallelogram and triangle of velocities; centre of gravity; logarithms; trigonometrical ratios and the solution of triangles.

After the exercises at the end of the book come two sets of four test papers, answers and index. R. C. L.

BOOKS RECEIVED FOR REVIEW.

J. B. Channon and **A. McL. Smith**. *Elementary aircraft mathematics. A revision course.* Pp. 97. 1s. 6d. 1942. (Longmans)

J. Douglas. *Survey of the theory of integration.* Pp. 47. 50 cents. 1941. (*Scripta Mathematica*, Yeshiva College, New York)

F. Franklin. *The four color problem.* Pp. 33. 50 cents. 1941. (*Scripta Mathematica*, Yeshiva College, New York)

A. Geary, H. V. Lowry and **H. A. Hayden**. *Mathematics for technical students. III.* Pp. viii, 395. 8s. 1942. (Longmans)

T. H. Ward Hill. *Practical mathematics for the A.T.C.* Pp. 120. 2s. 6d. 1941. (Harrap)

L. Infeld. *The fourth dimension and relativity.* Pp. 10. 25 cents. 1941. (*Scripta Mathematica*, Yeshiva College, New York)

C. J. Keyser. *Charles Sanders Peirce as a pioneer.* Pp. 24. 25 cents. 1941. (*Scripta Mathematica*, Yeshiva College, New York)

T. K. Raghavachari. *An introduction to analytical geometry and calculus.* Pp. xx, 192. Rs. 2. 1941. (Oxford University Press, Indian Branch)

A. Wintner. *The analytical foundations of celestial mechanics.* Pp. xii, 448. 36s. 1941. Princeton Mathematical Series, 5. (Princeton University Press; Humphrey Milford)

The development of the sciences. Second series. Edited by L. L. Woodruff. Pp. vii, 336. 18s. 6d. 1941. (Yale University Press; Humphrey Milford)

Lectures in topology. Edited by R. L. Wilder and W. L. Ayres. Pp. vii, 316. $3. 1941. (University of Michigan Press; Humphrey Milford)

The second yearbook of research and statistical methodology. Books and reviews. Edited by O. K. Buros. Pp. xx, 383. $5. 1941. (Gryphon Press, Higland Park, N.J.)

Mathematical Monographs. I. Edited by D. R. Curtiss. Pp. vii, 172. $2.25. 1941. Northwestern University Studies in Mathematics and the Physical Sciences, I. (Northwestern University, Evanston and Chicago)

The Philosophy of Alfred North Whitehead. Edited by P. A. Schilpp. Pp. xx, 745. $4. 1941. Library of Living Philosophers, 3. (Northwestern University, Evanston and Chicago)

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